Strong interaction of solitary waves for the fmKdV equation

Frédéric Valet, joint works with Arnaud Eychenne

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1. Introduction to the fmKdV equation
2. Solitary waves
   - Ground state
   - Multi-solitary waves
   - Previous results
   - Main theorem
3. Proof of the Main theorem
   - Asymptotic expansion
   - Open questions

With a local dispersion:

$$\partial_t u + \partial_x \left(\Delta u + u^3\right) = 0, \quad u : I_t \times \mathbb{R}_x \to \mathbb{R}. \tag{mKdV}$$

With a non- local dispersion:

$$\partial_t u + \partial_x \left(-|D|^\alpha u + u^3 \right) = 0, \quad u : I_t \times \mathbb{R}_x \to \mathbb{R}, \qquad \text{(fmKdV)}$$

with $\mathcal{F}\left(-|D|^\alpha f\right)(\xi) := -|\xi|^\alpha \mathcal{F}(u)$, and $1 < \alpha < 2$.

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Scaling, leaving the set of solutions invariant:

$$u \mapsto u_\lambda, \quad u_\lambda(t,x) = \lambda^{\frac{\alpha}{2(1+\alpha)}} u\left(\lambda t, \lambda^{\frac{1}{1+\alpha}} x\right).$$

L^2-subcritical. Conserved quantities:

$$M(u) = \int \frac{u^2(t)}{2}, \quad E(u) = \int \frac{1}{2}\left(|D|^{\frac{\alpha}{2}} u\right)^2 - \frac{1}{4} u^4$$

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For (fKdV), $\alpha \in [-1, 1]$ [Molinet-Pilod-Vento, 2018; Riaño, 2020]

Solitary waves of velocity $c > 0$ and shift $y \in \mathbb{R}$ the form $(t, x) \mapsto Q_c(x - ct - y)$
In the previous form, Q_c obeys the following equation:

$$-|D|^{\alpha} Q_c - cQ_c + Q_c^3 = 0.$$

- existence of solutions [Weinstein, 1985; Albert, Bona, Saut 1997]
- uniqueness of the ground state [Frank, Lenzmann 2013]; we denote it by Q_c

Periodic case of (fmKdV) [Natali, Le, Pelinovski, 2022]

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Stability [Angulo Pava 2018]: The solitary waves associated with the ground-states Q_c are orbitally stable in $H^{\frac{\alpha}{2}}$.

Definition

A multi-solitary wave u is a solution of (fKdV) which in large time is close to a sum of K decoupled solitons. More precisely, there exists $0 < c_1 < \cdots < c_K$, $T_0 > 0$, $C > 0$, *and* K *functions* $\rho_1, \cdots, \rho_K \in \mathcal{C}^1([T_0, +\infty), \mathbb{R})$ *such that* $\forall t \geq T_0$,

$$\left\| u(t) - \sum_{k=1}^{K} Q_{c_k}(\cdot - \rho_k(t)) \right\|_{H^{\frac{\alpha}{2}}} \leq \frac{C}{t^{\frac{\alpha}{2}}},$$

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Theorem (Eychenne, 2021)

Let us fix $K \in \mathbb{N}$ distinct velocities $0 < c_1 < \cdots < c_K$. There exists a multi-solitary wave u of (fKdV) associated to those previous velocities.

For subcritical (gKdV) (it includes (mKdV)!)

$$\partial_t u + \partial_x \left(\partial_x^2 u + |u|^{p-1} u\right) = 0, \quad p \in (2,5),$$

[Nguyen 17] : strong interaction between the solitons; there exists a solution u satisfying:

$$\left\| u(t,\cdot) - \sum_{i=1}^{2} (-1)^i Q\left(\cdot - t + (-1)^i c_0 \ln(c_1 t)\right) \right\|_{H^1} \to 0,$$

as $t \to +\infty$.

Theorem (Eychenne, V., preprint 2022)

There exists $T_0 > 0$, a solution u of (fmKdV) on $[T_0, +\infty)$ which behaves in large time as a sum of two strongly interacting solitary waves:

$$\lim_{t \to +\infty} \left\| u(t) - \sum_{k=1}^{2} (-1)^k Q\left(\cdot - \rho_k(t)\right) \right\|_{H^{\frac{\alpha}{2}}} = 0,$$

with, for a certain constant $c_0 > 0$:

$$\lim_{t \to +\infty} \left| \rho_k(t) - t + (-1)^k c_0 t^{\frac{2}{\alpha+3}} \right| = 0.$$

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- Accurate profiles $V = -R_1 + R_2 + \epsilon$:

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- Need the asymptotic expansion $Q(x) \sim_{+\infty} cx^{-\alpha-1} - \dots$. To inverse, need of orthogonality conditions. Thus the need to define:

$$W(t,x) \simeq \int_x^{+\infty} (|D|^\alpha + 1)^{-1} \left(\Lambda R_1(t,y) - \Lambda R_2\right) dy.$$

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Proposition (Eychenne, V., preprint, 2022)

Asymptotic of the ground-state at $+\infty$, with $a_0 > 0$:

$$Q(x) = \frac{a_0}{x^{\alpha+1}} + \frac{a_1}{x^{2\alpha+1}} + O_{+\infty}\left(\frac{1}{x^{\alpha+3}}\right),$$

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Ideas of the proof: [Bona, Li 1996]

$$Q = k \star Q^3, \quad \text{with} \quad k(x) = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^\alpha}\right)$$

Asymptotic expansion of:

$$k(x) = \frac{1}{\pi}\int_0^\infty \frac{e^{-s}}{s^{\frac{1}{\alpha}}} h\left(\frac{x}{s^{\frac{1}{\alpha}}}\right) ds, \quad h(y) = \int_0^\infty \cos(y\eta)\, e^{-\eta^\alpha}\, d\eta.$$

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Proof uses [Pólya, 1923]

Come back on the previous proof:

- bootstrap with an adequate functional on $u = V + \epsilon$, with adequate weights:

$$\mathcal{F}(t) \simeq \int \left(\frac{1}{2}|D|^\alpha \epsilon\epsilon + \frac{\epsilon^2}{2} - \frac{3}{2}V^2\epsilon^2 \right) \phi_1(t,x) + ...$$

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- Pseudo-differential operators. For instance : $\alpha \in (1,2)$:

$$\left\| \left[|D|^{\alpha}, \sqrt{|\phi'|} \right] u \right\|_2^2 \leq C \int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 \right) |\phi'| .$$

- Bootstrap argument on μ_1, μ_2, ϵ; topological argument for $\dot{z}_1$, $\dot{z}_2$.

- Collision of two solitons : is it elastic?

Thank you!