The distribution of the maximum protection number in random trees

Analytic and Probabilistic Combinatorics Workshop BIRS

Joint work with Clemens Heuberger and Stephan Wagner

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Modeling - Analysis - Optimization
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Bootstrapping and Double-Exponential Limit Laws (Prodinger & Wagner, 2015)

"It is a very typical situation that an extremal parameter in a combinatorial structure follows a discrete double-exponential distribution, and that fluctuations in the average occur."

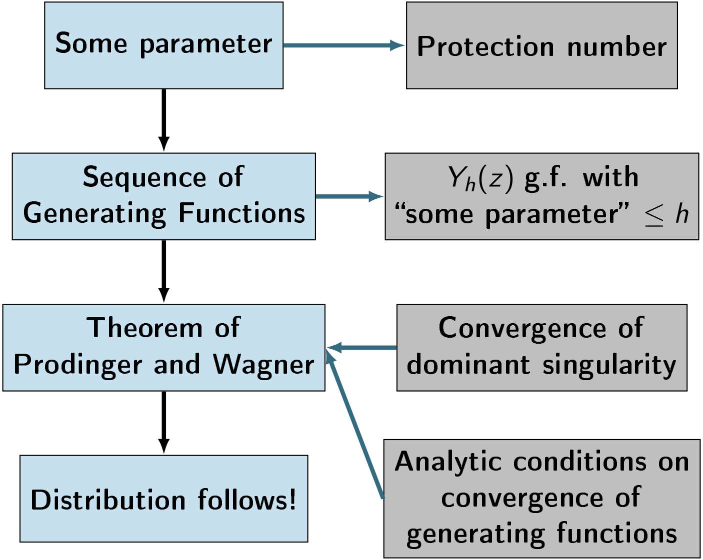
Some examples:

- Longest sequence of 1's in a 0-1 sequence.
- Longest sequence of $a \in \mathcal{A}$, $\mathcal{A}$ is an alphabet of size k.
- Longest horizontal segment in a Motzkin path.
- Maximum outdegree in planted plane trees.

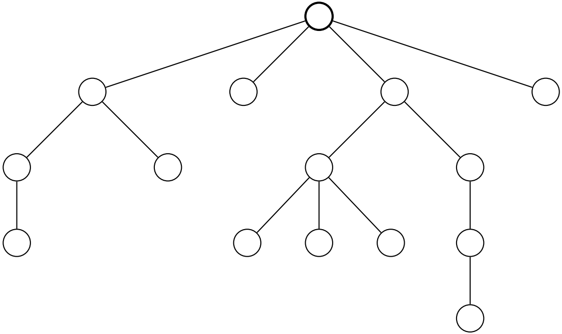
$$\mathbb{P}(X_n \leq h) = \exp(-An\rho^h)(1 + o(1)),$$

$$\mathbb{E}(X_n) = \log_b n + \log_b A + \frac{\gamma}{\log b} + \frac{1}{2} + \phi_b(\log_b An) + o(1).$$

Summary


Some parameter
Protection number
Sequence of Generating Functions
$Y_h(z)$ g.f. with "some parameter" $\leq h$
Theorem of Prodinger and Wagner
Convergence of dominant singularity
Analytic conditions on convergence of generating functions
Distribution follows!

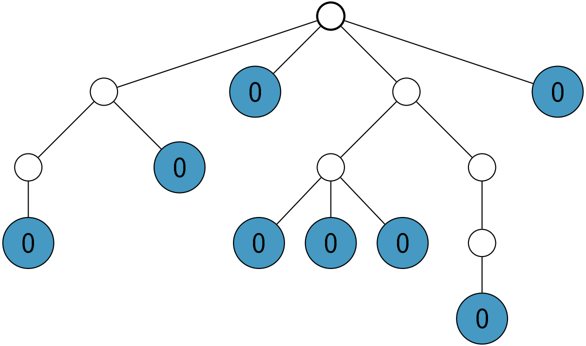
The protection number of a vertex



Definition

The protection number of a vertex v is the length of the shortest path from v to any leaf contained in the maximal subtree where v is a root.

The protection number of a vertex



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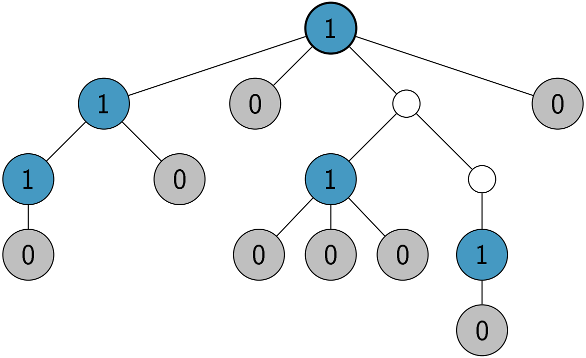

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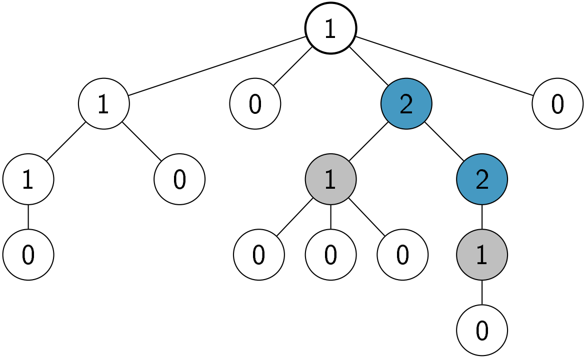

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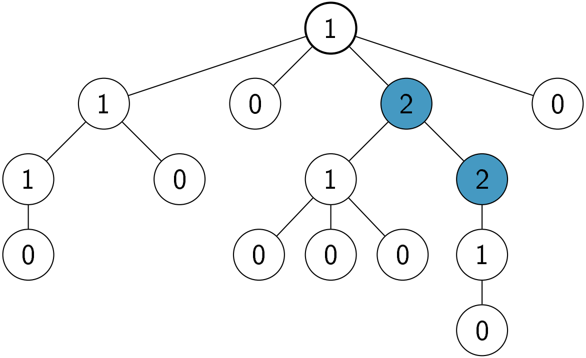

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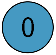
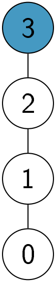
Maximum protection number: Some examples

- A maximum protection number of 0 means the tree is a single vertex.



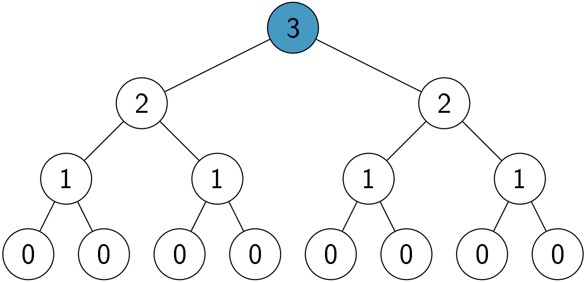
Maximum protection number: Some examples

- A maximum protection number of 0 means the tree is a single vertex.
- Paths (vertices are a leaf or have exactly one child) have a very high ratio of protection number to number of vertices.



Maximum protection number: Some examples

- A maximum protection number of 0 means the tree is a single vertex.
- Paths (vertices are a leaf or have exactly one child) have a very high ratio of protection number to number of vertices.
- Trees where vertices generally have more than one child have a low ratio of protection number to number of vertices.



Timeline of work on protection number of trees

- Number of vertices with protection number at least 2:
  - in ordered trees. *Cheon and Shapiro (2008).*
  - in k-ary trees, digital search trees, binary search trees, tries and suffix trees, random recursive trees.
    Devroye, Du, Gaither, Holmgren, Homma, Janson, Mahmoud, Mansour, Prodinger, Sellke, Ward (2010–2015).
- Number of vertices with protection number at least k, again in various types of trees.
  Bóna, Copenhaver, Devroye, Heuberger, Janson, Prodinger, Pittel (2014–2017).
- Protection number of the root. Plane trees, simply generated trees, Pólya trees.
  Gittenberger, Gołębiewski, Heuberger, Klimczak, Larcher, Prodinger, Sulkowska (2017–2021).

Simply generated trees

Definition

A simply generated tree has a generating function Y which satisfies the functional equation $Y(x) = x\Phi(Y(x))$ where Φ is a weight generating function $\Phi(x) = \sum_{n\geq 0} w_n x^n$, $w_n \geq 0$.

- Complete binary trees: $B(x) = x + xB(x)^2 = x(1+B(x)^2)$.
- Plane trees: $P(x) = x + xP(x) + xP(x)^2 + \cdots = x\frac{1}{1-P(x)}$.

Some things to note:

- $w_n \neq 0$ means the tree **can** have vertices with exactly n children.
- ρ is the (finite) radius of convergence or dominant singularity of $Y(x)$.
- $\tau = Y(\rho)$, so that $\Phi(\tau) = \tau\Phi'(\tau)$ and $\rho = \tau/\Phi(\tau) = 1/\Phi'(\tau)$.

Protection number of simply generated trees

Let $Y_{h,k}$ be the generating function for simply generated trees with:

- the maximum protection number of any vertex is $\leq h$,
- the protection number of the root is $\geq k$.

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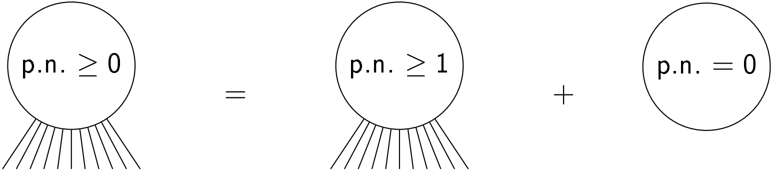
$$Y_{h,0}(x) = Y_{h,1}(x) + x,$$



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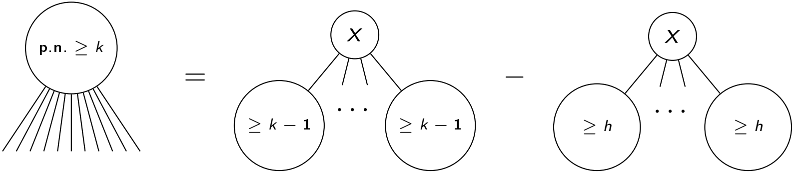
$$
\begin{aligned}
Y_{h,0}(x) &= Y_{h,1}(x) + x, \\
Y_{h,k}(x) &= x\Phi(Y_{h,k-1}(x)) - x\Phi(Y_{h,h}(x)), \qquad 1 \leq k \leq h.
\end{aligned}
$$



Protection number of simply generated trees

The system of functional equations:

$$\begin{aligned} Y_{h,0}(x) &= Y_{h,1}(x) + x, \\ Y_{h,k}(x) &= x\Phi(Y_{h,k-1}(x)) - x\Phi(Y_{h,h}(x)), \qquad 1 \leq k \leq h. \end{aligned}$$

We set $x := \rho_h$ (common radius of convergence of system for fixed h) and $\eta_{h,k} := Y_{h,k}(\rho_h)$, so the system becomes

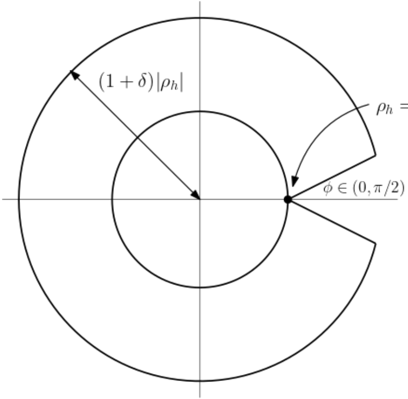
$$\begin{aligned} \eta_{h,0} &= \eta_{h,1} + \rho_h, \\ \eta_{h,k} &= \rho_h\Phi(\eta_{h,k-1}) - \rho_h\Phi(\eta_{h,h}), \qquad 1 \leq k \leq h. \end{aligned}$$

Determinant of Jacobian:

$$0 = \prod_{j=1}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) + \left(1 - \rho_h \Phi'(\eta_{h,0})\right)\left(1 + \sum_{k=2}^{h} \prod_{j=k}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right)\right).$$

Theorem of Prodinger and Wagner

$$Y_h(z) = \sum_{n \geq 0} y_{h,n} z^n \to Y(z)$$



$\rho_h = \rho + c\zeta^h + o(\zeta^h), \zeta \in (0, 1)$

$Y_h(z) = A_h(z) + C_h(1 - z/\rho_h)^\alpha + o((1 - z/\rho_h)^\alpha)$
uniformly in h

$Y(z) = A(z) + C(1 - z/\rho)^\alpha + o((1 - z/\rho)^\alpha)$

For details: Helmut Prodinger and Stephan Wagner. Bootstrapping and double-exponential limit laws. DMTCS, 2015.

Goal: Apply the Theorem of Prodinger and Wagner

Problem 1

Show that the dominant singularity for $Y_{h,0}$ is $\rho_h \in \mathbb{R}$, where

$$\rho_h = \rho + c\zeta^h + o(\zeta^h)$$

as $h \to \infty$ for some constants $\rho > 0$, $c > 0$ and $0 < \zeta < 1$.

The system that we must use to obtain this result is the following:

$$\eta_{h,0} = \eta_{h,1} + \rho_h,$$

$$\eta_{h,k} = \rho_h \Phi(\eta_{h,k-1}) - \rho_h \Phi(\eta_{h,h})$$

$$0 = \prod_{j=1}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) + \left(1 - \rho_h \Phi'(\eta_{h,0})\right)\left(1 + \sum_{k=2}^{h} \prod_{j=k}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right)\right).$$

Aim: $\rho_h = \rho + c\zeta^h + o(\zeta^h)$

Show:

1. $\rho_h \to \rho$.

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Show:

1. $\rho_h \to \rho$.
2. $\eta_{h,k} \to \eta_k$.
3. $\prod_{j=1}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) = O((\rho\Phi'(0))^h)$ and

$$1 + \sum_{k=2}^{h} \prod_{j=k}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) \to \frac{1}{1 - \rho\Phi'(0)}.$$

Aim: $\rho_h = \rho + c\zeta^h + o(\zeta^h)$

Show:

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4. $\eta_{h,0} = \tau + O({B_1}^h)$ and $\rho_h = \rho + O({B_1}^h)$.

Aim: $\rho_h = \rho + c\zeta^h + o(\zeta^h)$

Show:

1. $\rho_h \to \rho$.
2. $\eta_{h,k} \to \eta_k$.
3. $\prod_{j=1}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) = O((\rho\Phi'(0))^h)$ and
$$1 + \sum_{k=2}^{h} \prod_{j=k}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) \to \frac{1}{1-\rho\Phi'(0)}.$$
4. $\eta_{h,0} = \tau + O({B_1}^h)$ and $\rho_h = \rho + O({B_1}^h)$.
5. $\prod_{j=1}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) = (\rho\Phi'(0))^h \lambda_2 (1 + O({B_2}^h))$ and
$$1 + \sum_{k=2}^{h} \prod_{j=k}^{h} \left(\rho_h \Phi'(\eta_{h,j})\right) = \frac{1}{1-\rho\Phi'(0)(1+O({B_3}^h))}.$$

Asymptotic behaviour of the singularity

Lemma (Heuberger, SJS, Wagner, 2022+)

As $h \to \infty$, we have that

$$\rho_h = \rho + \frac{1}{\Phi(\tau)}(\rho\Phi'(0))^{h+1}\lambda_1(1-\rho\Phi'(0)) + o((\rho\Phi'(0))^h),$$

where

$$\lambda_1 = \eta_0 \prod_{i\geq 1} \frac{\eta_i}{\rho\Phi'(0)\eta_{i-1}}.$$

Asymptotic behaviour of the singularity

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where

$$\lambda_1 = \eta_0 \prod_{i \geq 1} \frac{\eta_i}{\rho\Phi'(0)\eta_{i-1}}.$$

With some additional analysis. . .

Result

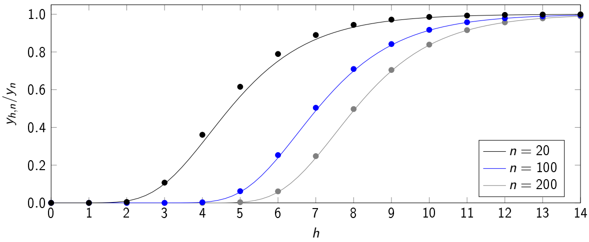
Theorem (Heuberger, SJS, Wagner, 2022+)

The probability that a random tree of size n has maximum protection number $\leq h$ is

$$\frac{y_{h,n}}{y_n} = exp\Big(-\frac{1}{\tau}\Phi'(0)\lambda_1(1-\rho\Phi'(0))n(\rho\Phi'(0))^h\Big)(1+o(1))$$

as $n \to \infty$ and $h = \log_{(\rho\Phi'(0))^{-1}} n + O(1)$.

Binary trees, $\Phi(x) = (1+x)^2$: Actual data (marks) plotted with the distribution (line)



Result

Theorem (Heuberger, SJS, Wagner, 2022+)

The expected value of the maximum protection number in a random tree of size n is

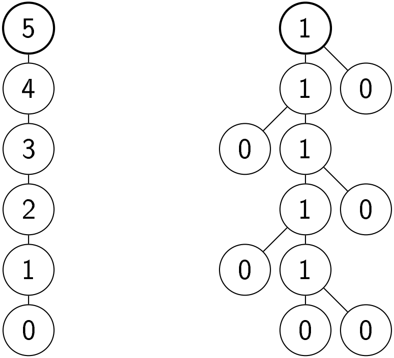
$$\log_b(n) + \log_b\Big(\frac{\lambda_2\Phi'(0)}{\Phi(\tau)}(1-\rho\Phi'(0))\Big) + \frac{\gamma}{\log(b)} + \frac{1}{2} + \psi_b\Big(\log_b\Big(\frac{\lambda_2\Phi'(0)}{\Phi(\tau)}(1-\rho\Phi'(0))n\Big)\Big) + o(1),$$

where γ denotes the Euler-Mascheroni constant, $\lambda_2 = \eta_0\prod_{i\geq 1}\frac{\eta_i}{\rho\Phi'(0)\eta_{i-1}}$, and ψ_b is the 1-periodic function that is defined by the Fourier series

$$\psi_b(x) = -\frac{1}{\log(b)}\sum_{k\neq 0}\Gamma\Big(-\frac{2k\pi i}{\log(b)}\Big)e^{2k\pi i x}.$$

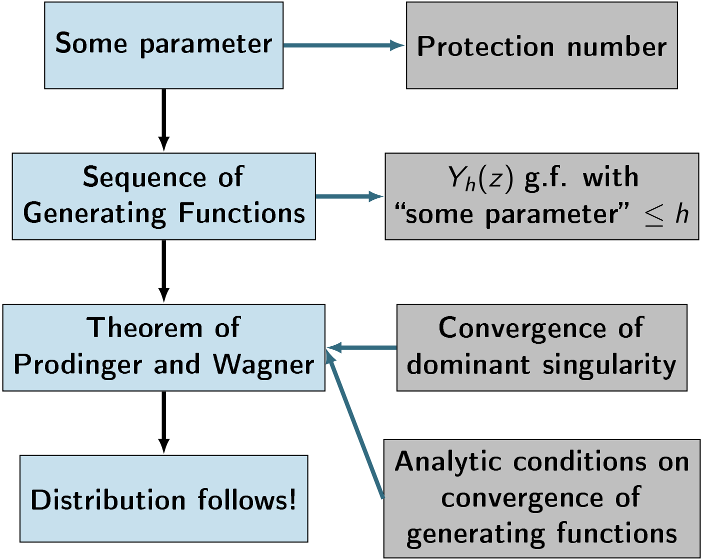
There's more!

Proofs and results depend on $\Phi'(0) \neq 0$. So we must consider the case where $\Phi'(0) = w_1 = 0$ separately.



- Set $r = \min\{s \in \mathbb{N} : \Phi^{(s)}(0) \neq 0\}$, $r \geq 2$.
- $\rho_h = \rho + c\zeta^{r^h} + o(\zeta^{r^h})$.

Thank you!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