Lower bounds on the amenability constant of the Fourier algebra

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Thanks to Brian, Volker and Keith

and

Happy retirement Tony!

In this talk, $\widehat{\otimes}$ denotes the projective tensor product of **Banach** spaces (not the operator space version).

Given a Banach algebra A, let us temporarily write $\mu : A \widehat{\otimes} A \to A$ for the bounded linear map satisfying

$$\mu(a \otimes b) = ab \qquad \text{for all } a, b \in A$$

and write $\kappa : A \to A^{**}$ for the natural embedding.

A virtual diagonal for A is some $\Delta \in (A \widehat{\otimes} A)^{**}$ such that $a \cdot \Delta = \Delta \cdot a$ and $\mu^{**}(\Delta) \cdot a = \kappa(a)$ for all $a \in A$. If A has a virtual diagonal, we say that it is amenable (Johnson, 1972).

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Example 1. Let A be any Banach algebra that is algebra-isomorphic to $\mathbb{C}^n$ with pointwise product. Then A has a (unique!) diagonal: this is the element of $A \otimes A$ corresponding to $\sum_{j=1}^{n} \delta_j \otimes \delta_j$. However, it is not clear how large the norm of this element is in $A \mathbin{\widehat{\otimes}} A$.

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Example 2. G a finite group; $A = \ell^1(G)$ with standard basis vectors $(e_g)_{g\in G}$, viewed as a Banach algebra with the convolution product. Then

$$\frac{1}{|G|} \sum_{g\in G} e_g \otimes e_{(g^{-1})}$$

is a diagonal element for A, which turns out to have norm 1 inside $A \widehat{\otimes} A$.

For a Banach algebra A, we define its amenability constant to be

$$\mathrm{AM}(A) := \inf \|\Delta\|_{(A\widehat{\otimes}A)^{**}}$$

where the infimum is over all virtual diagonals for A. We adopt the convention that $\mathrm{AM}(A) := +\infty$ when A is not amenable.

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Example 3. For any finite group G, we saw an explicit witness that $\mathrm{AM}(\ell^1(G)) = 1$. In fact, $\mathrm{AM}(L^1(G)) = 1$ for every amenable locally compact group G (Stokke, 2004).

So for L^1-group algebras there is a dichotomy: the amenability constant is either 1 or $+\infty$. This is very much not true when we work with Fourier algebras (unless we switch to discussing operator (space) amenabiity).

Let G be a **finite** group. Given $f \in \mathbb{C}^G$, and a representation $\sigma : G \to \mathcal{U}(H_\sigma)$, we define

$$\sigma(f) := \sum_{x \in G} f(x)\sigma(x) \in \mathcal{B}(H_\sigma).$$

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We then define $\mathrm{A}(G)$ to be $\mathbb{C}^G$ equipped with the following norm:

$$\|f\|_{\mathrm{A}} := \sum_{\pi \in \widehat{G}} \frac{d_\pi}{|G|} \|\pi(f)\|_{(1)}$$

where $\|\cdot\|_{(1)}$ is the trace-class norm.

Although the norm on $\mathrm{A}(G) \widehat{\otimes} \mathrm{A}(G)$ is hard to work with, there is a remarkable exact formula for $\mathrm{AM}(\mathrm{A}(G))$ when G is finite.

Theorem (Johnson, 1994)

Let G be a finite group. Then $\mathrm{AM}(\mathrm{A}(G)) = \dfrac{1}{|G|} \sum_{\pi \in \widehat{G}} (d_\pi)^3$.

Corollary

If G is a finite abelian group then $\mathrm{AM}(\mathrm{A}(G)) = 1$. If G is a finite non-abelian group, then $\mathrm{AM}(\mathrm{A}(G)) \geq 3/2$.

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Sketch of the proof of the 2nd part

Suppose G is finite and non-abelian, and let L be the set of 1-dimensional irreps of G. Since $d_\pi \geq 2$ for all $\pi \in \widehat{G} \setminus L$,

$$\mathrm{AM}(\mathrm{A}(G)) + \frac{|L|}{|G|} \geq \frac{1}{|G|}\sum_{\pi\in\widehat{G}} 2(d_\pi)^2 = 2.$$

But $|G|/|L|$ is the size of the derived subgroup of G, so must be ≥ 2. $\square$

For any locally compact group G, one can define its Fourier algebra $\mathrm{A}(G)$, in a way that extends the definition for finite groups.

The locally compact abelian setting

When G is a LCA group, it has a dual group $\widehat{G}$, and then $\mathrm{A}(G)$ is the range of the Gelfand/Fourier transform $L^1(\widehat{G}) \to C_0(G)$.

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Warning

The canonical map $\mathrm{A}(G_1) \mathbin{\widehat{\otimes}} \mathrm{A}(G_2) \to \mathrm{A}(G_1 \times G_2)$ has dense range, but is usually not an isometry.

In fact: for this comparison map to be surjective, either G_1 or G_2 must be virtually abelian (i.e. have an abelian subgroup of finite index).

Some notation

For a group G, define

$$\mathsf{adiag}(G) = \{(x, x^{-1}) \colon x \in G\} = \{(x, y) \in G \times G \colon xy = e\}.$$

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2. *the check map is completely bounded $\mathrm{A}(G_d) \to \mathrm{A}(G_d)$;*
3. G_d *(and hence G) is virtually abelian.*

The first part can be refined to give a quantitative statement:

Lemma (Runde, 2006)

Let G be a locally compact group. Then

$$\mathrm{AM}(\mathrm{A}(G)) \geq \left\|1_{\mathsf{adiag}(G)}\right\|_{\mathrm{B}}$$

where $\|\cdot\|_{\mathrm{B}}$ denotes the norm in $\mathrm{B}(G_d \times G_d)$.

Lemma (Runde, 2006)

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Quote from [Run06]

> *It remains to be seen whether or not Lemma 3.1 will eventually lead to a more satisfactory bound from below for the amenability constant of a Fourier algebra: very little seems to be known on the norms of idempotents in Fourier–Stieltjes algebras.*

The point of this talk: we can say quite a bit about the norm of **this particular idempotent!**

If Λ is a discrete group, we introduce the notation

$$\mathrm{AD}(\Lambda) := \left\| 1_{\mathsf{adiag}(\Lambda)} \right\|_{\mathrm{B}}$$

Reminder: $\mathrm{AM}(\mathrm{A}(G)) \geq \mathrm{AD}(G_d)$.

AD is an intrinsic invariant of a virtually abelian group, and we believe it deserves further study (regardless of the connection to amenability of Banach algebras). It has several useful hereditary properties.

- If H is a subgroup of G then $\mathrm{AD}(H) \leq \mathrm{AD}(G)$.
- $\mathrm{AD}(G_1 \times G_2) = \mathrm{AD}(G_1)\,\mathrm{AD}(G_2)$.
- For any G, there is a countable subgroup Λ such that $\mathrm{AD}(\Lambda) = \mathrm{AD}(G)$.

Let G be a finite group. Then $\mathrm{AD}(G) = \left\|1_{\mathsf{adiag}(G)}\right\|_{\mathrm{A}}$ and

$$\left\|1_{\mathsf{adiag}(G)}\right\|_{\mathrm{A}} = \frac{1}{|G \times G|} \sum_{\pi,\sigma \in \widehat{G}} d_\pi d_\sigma \left\|(\pi \otimes \sigma)(1_{\mathsf{adiag}(G)})\right\|_{(1)}$$

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For a Hilbert space H let X_{H} be the "flip map" on $\mathsf{H} \otimes_2 \mathsf{H}$.

Proposition

Let G be a finite group and let $\pi, \sigma \in \widehat{G}$. Then

$$(\pi \otimes \sigma)(1_{\mathsf{adiag}}(G)) = \begin{cases} 0 & \text{if } \pi \not\sim \sigma \\ \dfrac{|G|}{d_\pi} \mathsf{X}_{(H_\pi)} & \text{if } \pi = \sigma \end{cases}$$

Corollary (C., submitted)

For G finite, $\mathrm{AD}(G) = \dfrac{1}{|G|} \displaystyle\sum_{\pi \in \widehat{G}} (d_\pi)^3 = \mathrm{AM}(\mathrm{A}(G))$.

Now let G be a countable virtually abelian group. (This implies that $\sup_{\pi \in \widehat{G}} d_\pi < \infty$.) Let ν be Plancherel measure on $\widehat{G}$, normalized so that

$$\sum_{x \in G} |f(x)|^2 = \int_{\widehat{G}} \left(\|\pi(f)\|_{(2)} \right)^2 \, d\nu(\pi) \qquad (f \in c_{00}(G))$$

where $\|\cdot\|_{(2)}$ is the Hilbert–Schmidt norm.

With this normalization, $1 = \int_{\widehat{G}} d_\pi \, d\nu(\pi)$.

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Theorem (C., submitted)

Let G, ν be as above. Then $\mathrm{AD}(G) = \int_{\widehat{G}} (d_\pi)^2 \, d\nu(\pi)$.

Note: if G is finite, then $\nu(\{\pi\}) = \dfrac{d_\pi}{|G|}$ and we recover our earlier result.

Ideas in the proof

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- By the previous remarks, $\mathrm{AD}(G) = \|F\|_{L^1(\mathsf{whatever})}$. $\square$

Technical details

Let μ denote the pushforward of ν under the diagonal embedding $\widehat{G} \to \widehat{G} \times \widehat{G}$. Then (in the sense of Radon–Nikodym derivatives)

$$\frac{dF}{d\mu}(\pi, \sigma) = \begin{cases} 0 & \text{if } \pi \not\sim \sigma \\ \mathsf{X}_{(H_\pi)} & \text{if } \pi = \sigma \end{cases}$$

where, as before, X denotes the flip on the square of a Hilbert space.

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Theorem (C.)

Let G be a locally compact VA group which is non-abelian. Then $\mathrm{AM}(\mathrm{A}(G)) \geq \mathrm{AD}(G_d) \geq 3/2$.

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Proof. There is a countable non-abelian subgroup $\Lambda \leq G$, which is also VA since G is. Since $\mathrm{AD}(G) \geq \mathrm{AD}(\Lambda)$ it suffices to show that $\mathrm{AD}(\Lambda) \geq 3/2$.

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Proof. There is a countable non-abelian subgroup $\Lambda \leq G$, which is also VA since G is. Since $\mathrm{AD}(G) \geq \mathrm{AD}(\Lambda)$ it suffices to show that $\mathrm{AD}(\Lambda) \geq 3/2$. This follows from our explicit formula for $\mathrm{AD}(\Lambda)$ and the following technical fact. $\square$

Lemma

Let Λ be a countable, non-abelian VA group with normalized Plancherel measure ν. Let $\Omega_1 = \{\pi \in \widehat{\Lambda} \colon d_\pi = 1\}$. Then $\nu(\Omega_1) \leq 1/2$.

In fact, if $\mathrm{AD}(\Lambda) = 3/2$ then $\nu(\Omega_1) = 1/2$ and every irrep of Λ has degree ≤ 2. Pursuing this observation further, we obtain a complete characterization of those non-abelian groups which achieve the lower bound on AD.

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Theorem (C., submitted)

Let G be a (virtually abelian) discrete group. Then $\mathrm{AD}(G) = 3/2 \iff |G : Z(G)| = 4$.

Finite groups with this property include the dihedral group and quaternion group of order 8. For an infinite example: take the integer Heisenberg group and quotient by a suitable subgroup of its centre.

Conjecture

$\mathrm{AM}(\mathrm{A}(G)) = \mathrm{AD}(G_d)$ for every locally compact group G.

Character-theoretic invariants of finite groups

What is the relationship of $\mathrm{AM}(\mathrm{A}(\cdot)) = \mathrm{AD}(\cdot)$ to more traditional invariants of finite groups?

Further gap results

Work in progress indicates that if G is finite and non-abelian and $\mathrm{AD}(G) > 3/2$ then $\mathrm{AD}(G) \geq 5/3$. Can we prove the same gap result for general (virtually abelian) groups?