A Launch Event for PIMS CRG: L-functions in Analytic Number Theory

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1 Overview of the Field

The subject area of this event was analytic number theory, which focuses on arithmetic questions through the lens of L-functions. These generating series encode arithmetic information and are connected to a host of other mathematical fields, such as algebraic number theory, harmonic analysis, Diophantine approximation, probability, representation theory, and computational number theory.

2 Recent Developments and Open Problems

The organizers of this event have been awarded PIMS funding for a Collaborative Research Group (CRG) on L-functions in Analytic Number Theory. This CRG focuses on three main themes: moments of L-functions and automorphic forms, explicit results in analytic number theory, and comparative prime number theory. These areas have witnessed a surge of activity in the last decade including but not limited to the work on bounded gaps between primes (Zhang [7], Maynard [4] and the Polymath8 project led by Tao [5]), multiplicative functions in short intervals (Matomäki–Radziwiłł [3]), the resolution of the Erdős discrepancy conjecture (Tao [6]), and the major breakthroughs resulting from the work of Keating and Snaith [2] and Conrey et al. [1] that used random matrix theory to derive conjectures for moments of L-functions.

The goals of this CRG include tackling a variety of central problems in analytic number theory and elevating a new generation of experts that is more representative of our diverse society. The long-standing open problems we aim to answer include establishing several conjectures on moments of L-functions and completely resolving the problem of finding primes between consecutive cubes (a variant of Legendre?s primes-between-squares conjecture). We also plan to provide a direct link between the size of prime number error terms and zeros of L-functions, giving strong evidence for an influential 1980 conjecture of Montgomery. To solve these problems we aim to use a broad-scale and curated collaborative approach that brings together experts and rising stars with diverse competencies.

3 Scientific Progress and Presentation Highlights

The CRG launch event took place in a hybrid format during the weekend of November 18–20 with an in-person location at BIRS. The purpose of this launch event was threefold: publicizing the CRG's scientific activities, organizing and planning future events, and soliciting feedback and suggestions from participants.

We started this event with a presentation delivered by Habiba Kadiri and Nathan Ng about the CRG scientific objectives and a calendar of events and activities for the three-year duration of the CRG. This presentation was followed by an icebreaker activity to get online and in-person participants talking to each other. Thereafter, the CRG leaders delivered three introductory lectures about the CRG scientific themes, namely moments of L-functions and automorphic forms, explicit results in analytic number theory, and comparative prime number theory. Each of these lectures was followed by a discussion session about the scientific activities we would like to plan and the open problems we would like tackle through this CRG. Participants were invited to identify the items in which they would like to be involved, and the discussions were geared towards determining areas of overlapping research interests, planning future visits and events, and launching cross-collaborations among participating faculty members, postdoctoral fellows and graduate students.

One of the highlights of this 2-day event was the interactive and thought-provoking session led by Martha Mathurin-Moe, the EDI executive director at the University of Lethbridge. The participants engaged in a challenging discussion about the importance of EDI work in mathematics, and they shared ideas for concrete practices to be incorporated into our EDI plan to overcome systemic barriers and ensure that our priorities of inclusion and support are achieved in all the activities to be organized.

The video recordings and slides for the CRG overview, the three scientific lectures, and the slides used for the EDI discussion are posted on the BIRS webpage and linked to on the event's webpage in the "After the event" section.

Speakers, Titles and Abstracts

Alia Hamieh (UNBC) and Nathan Ng (University of Lethbridge)
Title: *Moments of L-functions and Automorphic Forms*

Results on the asymptotic behaviour of moments of L-functions have deep implications regarding their size, their zero distribution, their value-distribution, and their nonvanishing at special points. Indeed, studying the moments of L-functions was first motivated by a conjecture of Lindelof on the size of L-functions in the critical strip. The $2k$-th moment of the Riemann zeta function is given by $I_k(T) = \int_0^T |\zeta(\frac{1}{2}+it)|^{2k}\, dt$. It is believed that $I_k(T) \sim c_k T(\log T)^{k^2}$ for positive real numbers k. This is known to be unconditionally true for $k = 1, 2$ where the constants c_1 and c_2 have been determined explicitly. A major breakthrough in analytic number theory happened in 1998 when Keating and Snaith conjectured precise values for the constants c_k based on considerations from random matrix theory. This was followed by the influential work of Conrey et al. in 2005 in which they developed more precise conjectural asymptotic formulae for integral moments of L-functions identifying lower order main terms.

In this talk, we survey results on moments of L-functions starting with the classical work of Hardy-Littlewood in 1918 on the asymptotic formula for the second moment of the Riemann zeta function. Moreover, we discuss developments pertaining to the discrete moments of the Riemann zeta function. We also give an overview of recent results on moments of L-functions associated with quadratic characters and automorphic forms. In our discussion, we highlight the important tools used in studying such moments including approximate functional equations, multiple Dirichlet series, random matrix theory, spectral theory of automorphic forms and shifted convolution sums. We are particularly interested in exploring connections between the multiple Dirichlet series approach and the approximate functional equation approach to studying moments of L-functions, perhaps opening up new ways to understanding some of these moments.

Habiba Kadiri (University of Lethbridge)
Title: *Explicit Number Theory*

This talk will be an overview of explicit results in number theory, starting with Rosser and Schoenfeld who, between 1939 and 1976, proved a series of theorems about the zeros of the Riemann zeta function and the error term in the prime number theorem. The rise of computational tools has allowed us to partially verify conjectures, such as the Riemann Hypothesis, and to establish or refine statements of conjectures, and has also helped to explicitly confirm some statements known to be true only asymptotically, such as the Odd Goldbach Conjecture. The most recent years have seen an exponential increase in results of an explicit nature, with various ?schools? essentially throughout Canada, US, Europe, and Australia, and with various objects of study, from primes and the Riemann zeta functions, to arithmetic progressions and Dirichlet L-functions, and to primes in number fields and Hecke L- or Dedekind zeta-functions. Explicit results are interesting on their own as they quantitatively measure the state of our understanding and the efficiency of our techniques. Their nature also allows a wide array of applications in Diophantine approximation, arithmetic, cryptography, and other fields of mathematics.

Greg Martin (UBC)
Title: *Comparative Prime Number Theory*

We will first summarize the significant results and functions of interest in comparative prime number theory. Starting with the error term in the prime number theorem (and the "races" between $\psi(x)$ and x and between $\pi(x)$ and $\mathrm{li}(x)$), we will continue with the Mertens sum and the problems of Plya and Turn, and then move to comparing counting functions for primes in arithmetic progression (the "race" between $\pi(x; q, a)$ and $\pi(x; q, b)$), where it is also natural to compare more than two functions. All of these quantities have explicit formulas involving the zeros of $\zeta(s)$ or $L(s, \chi)$, and we describe how to derive (usually conditionally) limiting distributions for the corresponding normalized error terms; from those limiting distributions we can obtain quantitative statements such as the logarithmic density of the set for which one such function is greater than another. Finally, we will propose some major directions of ongoing and future research, including generalizations to function fields, the frequency of lead changes in these races, correlations among these normalized error terms, and the possible consequences of a quantitative linear-independence statement for the imaginary parts of the zeros of $\zeta(s)$.

Martha Mathurin-Moe (University of Lethbridge)
Title: *EDI and Mathematics: Disruption and Opportunity*

Does Equity, Diversity, and Inclusion (EDI) really work? This is a question that many persons have asked. But the challenge with EDI work is while the field is constantly evolving there are a lot of fears or misconceptions that further complicates this work. Mathematics is no exception. The Science, Technology, Engineering and Mathematics (STEM) fields continue to see huge underrepresentation by women and racialized groups (Statistics Canada, 2019). In fact, the research has shown that the playing field has not always been leveled and there continues to be systems and structures that exclude Indigenous, Black, Brown, and racialized bodies from academic spaces (Joseph-Salisbury, 2019; Lopez & Jean-Marie, 2021). So, although we speak about a post racial era, the events of 2020 have pushed to the forefront the continued harsh realities faced by marginalized bodies within the social and academic discourse.
In this session, we will attempt to critically examine and unpack the importance of EDI work within the mathematics ecosystem, disrupt and open up a space for brave conversation that challenges the denial of racial and social issues within academia (Kappler, 2020; Kendi, 2019). Most importantly we will attempt to examine key practical strategies that can be done to address institutional and structural inequities within the discipline of mathematics. Racial and social inequalities are a policy and power issue that must be addressed by interrogating the systems and structures that continue to uphold these ideals (Kendi, 2019). There is journey ahead, and mathematicians have the opportunity to reshape and redefine its research, teaching, and scholarship ecosystem.

4 Impact of the Hybrid Format

It is often challenging for researchers from further geographical regions to justify the time commitment and travel expenses associated with attending a 2-day event in Banff. This is especially true for researchers with limited travel support, those who might face challenges or delays in obtaining visas, those with more caregiving duties, or those with health restrictions. The hybrid nature of the launch event allowed us to make it more accessible to a broader pool of participants and to bring together world experts and rising stars in analytic number theory with diverse competencies to contribute to the CRG plans. Eventually, 30 of the 48 attendees participated virtually from all over the world. The availability of excellent IT support provided by BIRS staff allowed for a successful hybrid experience despite few tehnical difficulties that we encountered along the way.

5 Demographics

We had 87 registered participants, among which 18 attended in person and 30 attended online.

- 16 of the 48 attendees were female.
- 25 attendees were professors, 10 were postdocs, 11 were graduate students, one was an undergraduate student, and one was a university employee.
- 28 attendees were from Canada, 8 were from the USA, and 12 were from outside North America.
- 46 attendees had a background in mathematics, one in statistics, and one in psychology.

6 Outcome of the Meeting

Thanks to its hybrid format, this event allowed us to launch our CRG activities with a broad and diverse audience. The three scientifc lectures were highly formative in summarizing the recent advances in three of the most active areas in analytic number theory. During the discussion sessions following the three scientific lectures, a wide range of state-of-the-art problems in these areas were described by the CRG leaders and new problems were suggested by the participants, thus expanding our problem list. Rather than solving these problems, the discussions facilitated by the CRG leaders were effective in determining common areas of research interests among participants, planning future visits and events, and launching cross-collaborations.

References

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