Cohomogeneity Two Manifolds of Positive Sectional Curvature

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Our group split our time at BIRS between two projects: the first was the cohomogeneity two project we described in our proposal, and the second was on estimating the lengths of geodesics.

1 Project 1: Cohomogeneity Two

Setup

A manifold admitting a group action with k-dimensional orbit space is a *cohomogeneity* k manifold. Manifolds of positive curvature admitting isometric group actions of cohomogeneity 0, that is homogeneous spaces, have been classified. (see Berger [3], Bérard-Bergery [2], Wallach [20], Aloff and Wallach [1], Wilking [21], and Wilking and Ziller [23]). A classification of cohomogeneity one manifolds of positive curvature was achieved in all dimensions except for 7, where a list of candidates have been given (see Searle [12], Verdiani [18, 19], and Grove, Wilking, and Ziller [10]). It is then natural to consider the problem of classifying simply-connected, closed, cohomogeneity two manifolds of positive curvature.

Let G be a compact, connected Lie group. Recall that an isometric G-action on M^n, a complete n-manifold, is called *polar* if it admits a *section*, that is, an immersed submanifold, Σ, of dimension equal to $\dim(M/G)$, that meets every orbit orthogonally. By work of Fang, Grove, and Thorbergsson [6] a closed, simply-connected, positively curved manifold admitting a polar cohomogeneity k action with $k \geq 2$ is equivariantly diffeomorphic to a compact rank one symmetric space (CROSS) with the corresponding polar action. Thus, in order to classify simply-connected, closed, cohomogeneity two manifolds of positive curvature, it remains to understand the case of non-polar actions.

Goals

The broad goal here is to classify such cohomogeneity two actions: that is, to find all possible M, up to diffeomorphism, and G, and to describe the action of G on M up to (equivariant) diffeomorphism. Coming into our stay at BIRS, our goal was to prove a classification theorem in low dimensions for closed, simply-connected, positively curved manifolds admitting an isometric, non-polar, cohomogeneity two action.

A useful tool for such actions is that of the *G-manifold reduction*, see Grove and Searle [9] and Skjelbred and Straume [16]. The idea is to reduce the G-action on M to the case of a *core group*, ${}_cG$, acting on a *core* *manifold*, ${}_cM$. The core group ${}_cG$ is defined to be $N_G(H)/H$, where $N_G(H)$ denotes the normalizer of H in G, the set of g such that $gHg^{-1} = H$. The core manifold ${}_cM$ is defined to be the closure of the set M^H of points in M fixed by (a particular copy of) H. The quotient ${}_cM/{}_cG$ is isometric to M/G, and the principal isotropy of the action of ${}_cG$ on ${}_cM$ is the identity only. Thus, the only properties of the original G-action that are not preserved in the reduction are that ${}_cM$ might not be simply connected and ${}_cG$ might not be connected. Note that the original action is polar if and only if the core group ${}_cG$ is finite. Using this technique, we have been able to prove the following proposition.

Proposition 1.1. *Let G be a compact, connected Lie group acting isometrically, effectively and by cohomogeneity two on M^n a closed, simply-connected manifold of positive curvature. Suppose further that the action is non-polar and that there is a G-manifold reduction of the G-action on M with core group of rank ≥ 2. Then M decomposes as a G-invariant union of disk bundles.*

The proof of this theorem relies on understanding the orbit space of the group action, M/G. In particular, M/G is homeomorphic to S^2 or D^2. If $M/G = S^2$, then there can be at most 2 isolated exceptional orbits, and if $M/G = D^2$, there are 4 possibilities: M/G has 0, 1, 2, or 3 vertices. In all but the case where $M/G = D^2$ and has 3 vertices, it is well-understood that M admits a G-invariant disk bundle decomposition. In the last case, in order to show that M admits a G-invariant disk bundle decomposition, it suffices to prove that two of the vertices of M/G are right angles.

This still leaves us to better understand non-polar G-actions on M with ${}_cG_0$, the connected component of ${}_cG$, is isomorphic to S^1. This case is well-understood when $M = S^n$, see Straume [14] and [15], but remains to be explored for all other such manifolds. While we have not encountered any examples of such manifolds that do *not* admit a G-invariant disk bundle decomposition, we also cannot rule out the possibility that such examples do exist.

We were able to make significant progress on the classification question. Going into our stay, we were not yet able to rule out the possibility that the Wu manifold, $SU(3)/SO(3)$, could admit such an action with a G-invariant metric of positive curvature in dimension 5, nor had we completed the classification in dimension 6. At the end of our stay, we were able to prove the following theorem.

Theorem 1.2. *Let G be a compact, connected Lie group acting isometrically, effectively and by cohomogeneity two on M^n a closed, simply-connected n-manifold of positive curvature. Suppose further that the action is non-polar, $n \leq 6$, and for $n = 5$, the action cannot be almost free. Then M^n is (equivariantly) diffeomorphic to S^n, $\mathbb{C}\mathrm{P}^{n/2}$, or $V^6 = SU(3)/T^2$.*

Observation 1.3. *The theorem is rigid: we can show that all such manifolds admit an isometric, non-polar, cohomogeneity two action.*

Observation 1.4. *The case where $n = 5$ and the action is free has recently been resolved by work of Cavenaghi, Grama, and Sperança [5], who have shown that there is no such action. Using the Connectedness Lemma of Wilking [22], one sees that if the action is almost free and the exceptional orbits are not isolated, that there is no such action. However, the cases where the action is almost free and has isolated exceptional orbits have yet to be understood. They have been studied by Simas [13] who showed that the only candidates are diffeomorphic to the two S^3 bundles over S^2.*

2 Project 2: Lengths of Geodesics

We note that this second project also includes Isabel Beach as a collaborator and will form part of the focus of our team at the Women in Geometry 3 workshop to be held in November 2023 at BIRS.

Setup

Let $\mathcal{M}^D_{k,v}(n)$ denote the set of n-dimensional closed Riemannian manifolds M, with sectional curvature bounded below by k, volume bounded below by v, and diameter bounded above by D. The property that distinguishes this class of manifolds is their uniform local contractibility: a result of Grove and Petersen in [8], states that there exist r and R, both depending on k, v, D, n, such that every ball of radius r is null-homotopic in the concentric ball of radius R.

A theorem of Serre states that in a closed Riemannian manifold M, given any two points p and q there are infinitely many geodesics from p to q. The goal of our project is to estimate the growth of the lengths of these geodesics, as M ranges over $\mathcal{M}_{k,v}^{D}(n)$. In particular, we have the following conjecture:

Conjecture 2.1. *Let $M \in \mathcal{M}_{k,v}^{D}(n)$ and assume M is simply connected. Then there is a constant $C(k,v,D,n)$, such that for every $\ell \in \mathbb{N}$, there are at least ℓ geodesics from p to q of length at most $C(k,v,D,n) \cdot \ell$.*

Existing results

Our project is based on results of Nabutovsky and Rotman [11]. In [11], they paper prove that for any points p and q in closed manifold M of diameter D, there are at least ℓ geodesics from p to q of length at most $4nD \cdot \ell^2$. In particular, the sequence of lengths in order grows at most quadratically with ℓ. The hope is that by introducing the bounds on curvature and volume, we can improve the bound so that the lengths grow linearly with ℓ.

Immediate goal

We have sketched a proof of our conjecture with an additional hypothesis, namely, we suppose there is a constant c such that all loops of length at most $2D$ are null-homotopic through loops of length at most cD. At BIRS we began the process of writing up this result. With this additional constraint, instead of a obtaining a constant $C(k,v,D,n)$, we get a constant $C(k,v,D,n,c)$ that depends on c as well. One special case is when there are no loops of length at most $2D$ that are local minima of length. In this case, every loop of length at most $2D$ is null-homotopic through a path of loops that never increases in length, so we have $c = 2$, and our constant depends only on k, v, D, n. We hope to prove the conjecture without this additional hypothesis, but so far do not have good techniques for eliminating it.

Next goal

As we worked on writing up our result, we also explored an extension of the problem, where instead of geodesics between two points we consider periodic geodesics. While a periodic geodesic is a closed loop that is geodesic all along the loop, a geodesic loop is a geodesic from a point p to itself, but the incoming and outgoing directions at p might have some angle between them. The work of Gromoll and Meyer [7] gives topological conditions on a closed manifold M that guarantee infinitely many periodic geodesics, and Sullivan and Vigué-Poirrier [17] show that these conditions are satisfied if the rational cohomology of M cannot be generated by a single generator. Under these hypotheses, it makes sense to ask about the growth of the length of these periodic geodesics, and we can hope that the conclusion of our theorem for geodesics is also true for periodic geodesics. When we count periodic geodesics, each periodic geodesic can be iterated and/or shifted to produce infinitely many other periodic geodesics; thus, we say that periodic geodesics are distinct only if their images are distinct loops in M.

What we hope to prove here is that if $M \in \mathcal{M}_{k,v}^{D}(n)$ and the rational cohomology of M cannot be generated by a single class, and all loops of length at most $2D$ are null-homotopic through loops of length at most cD, then for all $\ell \in \mathbb{N}$, there are at least ℓ geometrically distinct periodic geodesics of length at most $C(k,v,D,n,c) \cdot \ell$. We spent some of our time at BIRS trying to understand the proof of Gromoll and Meyer, which is based on careful analysis of index and nullity of a periodic geodesic and its iterates based on paper [4] of Bott, to see how easily it could be made quantitative in this way.

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