Differential Algebraic Generating Series for Walks in the Quarter Plane

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Lattice Paths, Combinatorics, and Interactions

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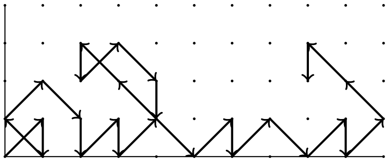
Walks

Consider the walks in the quarter plane starting from $(0,0)$ with steps in a fixed set

$$\mathcal{D} \subset \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\} \leftrightarrow \{(i,j) \mid i,j \in \{-1,0,1\}\}$$

Example with possible directions

$$\mathcal{D} = \{\nwarrow, \nearrow, \searrow, \downarrow\}.$$



Assign probabilities $d_{i,j}$ to each $(i,j) \in \mathcal{D}$ and ask what is

$$\mathbb{P}\left((0,0) \to^k (l,s)\right)$$

the probability that a walk starting at $(0,0)$ ending at (l,s) after k steps?

Models and Generating Series

Weighted Model: Fix a set of probabilistic weights

$$\mathcal{W} = \{(d_{i,j})_{i,j=-1,0,1} \in (\mathbb{Q} \cap [0,1])^9 \text{ with } \sum d_{i,j} = 1\},$$

associated with a set of directions $\mathcal{D} := \{(i,j) | d_{i,j} \neq 0\}$

Unweighted Model: the $d_{i,j} = \frac{1}{|\mathcal{D}|}$ for all $(i,j) \in \mathcal{D}$ and $d_{0,0} = 0$. In this case

$$\mathbb{P}\left((0,0) \rightarrow^k (l,s)\right) = \frac{\#(\text{walks from } (0,0) \text{ to } (l,s) \text{ with } k \text{ steps})}{|\mathcal{D}|^k}$$

Generating Series: Fix $\mathcal{W}$ (and therefore $\mathcal{D}$)

$$Q_{\mathcal{W}}(x,y,t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \rightarrow^k (l,s)\right) x^l y^s t^k$$

converges for $|x|, |y| \leq 1$ and $|t| < 1$.

Classification

$$\text{Algebraic/Analytic properties of } Q_{\mathcal{W}}(x,y,t)$$
$$\Updownarrow$$
$$\text{Asymptotic properties of } \mathbb{P}\left((0,0) \rightarrow^{k} (l,s)\right)$$

Classification problem: when is $Q_{\mathcal{D}}(x,y,t)$

- Algebraic over $\mathbb{C}(x,y,t)$?
- Holonomic over $\mathbb{C}(x,y,t)$? (x-, y-, and t-holonomic)
- Differentially Algebraic over $\mathbb{C}(x,y,t)$? (x-,y-, and t-diff. algebraic)

$f(x,y,t)$ is <u>x-holonomic</u> if for some n and $a_i \in \mathbb{C}(x,y,t)$,

$$a_n(x,y,t)\frac{\partial^n f}{\partial x^n} + \ldots + a_1(x,y,t)\frac{\partial f}{\partial x} + a_0(x,y,t)f = 0$$

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- Differentially Algebraic over $\mathbb{C}(x,y,t)$? (x-,y-, and t-diff. algebraic)

$f(x,y,t)$ is x-differentially algebraic if for some n and polynomial $P \neq 0$,

$$P(x,y,t,f,\frac{\partial f}{\partial x},\ldots,\frac{\partial^n f}{\partial x^n}) = 0$$

Classification

Fayolle, Iasnogorodski, Malyshev (1999), Bousquet-Mélou, Mishna (2010) - associate to a model $\mathcal{W}$,

- an algebraic curve $E_{\mathcal{W}}$ of genus 0 or 1, and
- a group $G_{\mathcal{W}}$, finite or infinite.

256 choices for $\mathcal{D}$ $\xrightarrow{\textit{triviallity}, \textit{symmetries}}$ 79 interesting ones

Results: For the **79** unweighted models

- $|G_{\mathcal{D}}| < \infty$ for **23** walks $\Rightarrow Q_{\mathcal{D}}(x, y, t)$ algebraic or holonomic.
  $\rightarrow$ A. Bostan, M. Bousquet-Mélou, M. van Hoeij, M. Kauers, M. Mishna, ...
- $|G_{\mathcal{D}}| = \infty$ for **56** walks $\Rightarrow Q_{\mathcal{D}}(x, y, t)$ **not** holonomic.
  - 5 walks with genus($E_{\mathcal{W}}$) $= 0$ $\rightarrow$ S. Melzcer, M. Mishna, A. Rechnitzer, ...
  - 51 walks with genus($E_{\mathcal{W}}$) $= 1$ $\rightarrow$A. Bostan, I. Kurkova, K. Raschel, B. Salvy, ...
- Differentially Algebraic???

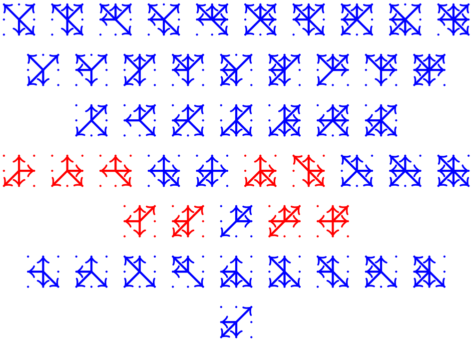
Classification

$$Q_{\mathcal{W}}(x,y,t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \to^k (l,s)\right) x^l y^s t^k$$

- $Q_{\mathcal{W}}(x,y,t)$ is x-DA $\Longleftrightarrow$ $Q_{\mathcal{W}}(x,0,t)$ is x-DA (similarly for y-DA)
- $Q_{\mathcal{W}}(x,0,t)$ is x-DA $\Longleftrightarrow$ $Q_{\mathcal{W}}(0,y,t)$ is y-DA.
- *(Dreyfus-Hardouin 2019, Dreyfus 2021)* $Q_{\mathcal{W}}(x,y,t)$ is NOT x-DA $\Longrightarrow$ $Q_{\mathcal{W}}(x,y,t)$ is NOT t-DA

Focus on y-DA-properties of $Q_{\mathcal{W}}(0,y,t)$

The 51 unweighted models with $|G_{\mathcal{D}}| = \infty$, genus$(E_{\mathcal{W}}) = 1$



Theorem (Dreyfus-Hardouin-Roques-S, 2018): For $t \in \mathbb{C} \backslash \overline{\mathbb{Q}}$

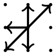
1. In 42 cases, $Q_{\mathcal{D}}(0, y, t)$ is not y-DA.
2. In 9 cases, $Q_{\mathcal{D}}(0, y, t)$ is y-DA but not holon.

- $Q_{\mathcal{D}}(x, y, t)$ are x-,y-, and t-DA in 9 cases first shown by O. Bernardi, M. Bousquet-Mélou, K. Raschel

What about weighted models?

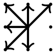
Examples

Ex.1 The weighted model



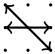
is differentially algebraic iff $d_{-1,-1}d_{1,1} - d_{0,-1}d_{0,1} = 0$

Ex.2 The weighted model



is differentially algebraic iff $d_{-1,1}d_{0,1}^2 - d_{0,1}d_{-1,-1}d_{0,-1} + d_{1,1}d_{-1,-1}^2 = 0$

Ex.3 The weighted model



is differentially algebraic iff $d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$. In this case the group is D_4 or D_8 and the generating series is holonomic.

- **Generalities about Walks:** Functional Equation, Curve, Group, Difference Equation

- **Theorems for Differential Algebraicity:** Galois Theory, Certificates, Orbit Residues

- **Algorithms for Differential Algebraicity:** Mordell-Weil Lattices, Néron-Tate Height

Generalities: Functional Equation of the Walk

Generating series: Fix $\mathcal{W}$ (and therefore $\mathcal{D}$)

$$Q_{\mathcal{W}}(x, y, t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \to^k (l, s)\right) x^l y^s t^k$$

Step Inventory: $\mathcal{S}_{\mathcal{W}}(x, y) = \sum_{(i,j)} d_{i,j} x^i y^j$
Kernel of the Walk: $K_{\mathcal{W}}(x, y, t) = xy(1 - t\mathcal{S}_{\mathcal{W}}(x, y))$ - biquadratic
Functional Equation:

$$\begin{aligned} K_{\mathcal{W}}(x, y, t) Q_{\mathcal{W}}(x, y, t) = xy \\ - K_{\mathcal{W}}(x, 0, t) Q_{\mathcal{W}}(x, 0, t) - K_{\mathcal{W}}(0, y, t) Q_{\mathcal{W}}(0, y, t) \\ + K_{\mathcal{W}}(0, 0, t) Q_{\mathcal{W}}(0, 0, t). \end{aligned}$$

Kernel Method: What hppens when $K_{\mathcal{W}}(x, y, t) = xy(1 - t\mathcal{S}_{\mathcal{W}}(x, y)) = 0$ tells us what happens in general.

Curve of the Walk

Step Inventory: $\mathcal{S}_\mathcal{W}(x,y) = \sum_{(i,j)\in\mathcal{W}} q_{i,j}x^iy^j$

Kernel of the Walk: $K_\mathcal{W}(x,y,t) = xy(1 - t\mathcal{S}_\mathcal{W}(x,y))$ - biquadratic

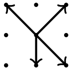
Functional Equation:

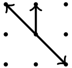
$$K_\mathcal{W}(x,y,t)Q_\mathcal{W}(x,y,t) = xy - K_\mathcal{W}(x,0,t)Q_\mathcal{W}(x,0,t) - K_\mathcal{W}(0,y,t)Q_\mathcal{W}(0,y,t) + K_\mathcal{W}(0,0,t)Q_\mathcal{W}(0,0,t).$$

Fix $t \in \mathbb{C}\backslash\overline{\mathbb{Q}}$. The Curve of the Walk is the curve

$$E_\mathcal{W} = \overline{\{(x,y) \mid K_\mathcal{W}(x,y,t) = 0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

Fact: $K_\mathcal{W}$ irreducible $\Rightarrow$ $E_\mathcal{W}$ has genus 0 or 1.

Ex: 1) $\mathcal{D} =$  $E_\mathcal{W} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \quad \Rightarrow g(E_\mathcal{W}) = 1$

2) $\mathcal{D} =$  $E_\mathcal{W} : xy - t(y^2 + xy^2 + x^2) = 0 \quad \Rightarrow g(E_\mathcal{W}) = 0$

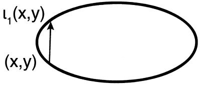
for $t \in \mathbb{C}\backslash\overline{\mathbb{Q}}$

Group of the Walk

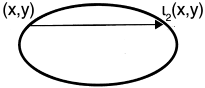
$$E_{\mathcal{W}} = \overline{\{(x,y) \mid K_{\mathcal{W}}(x,y,t)=0\}}^{\textit{Zariski}} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

We define two involutions of $E_{\mathcal{W}}$ and an automorphism:

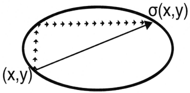
$$\iota_1(x,y) = \left(x, \frac{1}{y}\frac{\sum_i q_{i,-1}x^i}{\sum_i q_{i,1}x^i}\right)$$



$$\iota_2(x,y) = \left(\frac{1}{x}\frac{\sum_j q_{-1,j}y^j}{\sum_j q_{1,j}y^j}, y\right)$$



$$\sigma_{\mathcal{W}} = \iota_2 \circ \iota_1$$



The Group of the Walk $G_{\mathcal{W}}$ is the group generated by ι_1, ι_2.

Facts: 1) $\sigma_{\mathcal{W}}$ is a QRT-map. (Duistermaat - *Discrete Integrable Systems*)
2) $G_{\mathcal{W}}$ is infinite iff $\sigma_{\mathcal{W}}$ is infinite.
3) $g(E_{\mathcal{W}}) = 0 \Rightarrow G_{\mathcal{W}}$ are fractional linear trans.
4) $g(E_{\mathcal{W}}) = 1 \Rightarrow \exists \mathbf{P} \in E_{\mathcal{W}}$, s.t. $\sigma_{\mathcal{W}}(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$.

Kernel Curve $E_{\mathcal{W}} = \overline{\{(x,y) \mid K_{\mathcal{W}}(x,y,t)=0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$

Group of the Walk $G_{\mathcal{W}}$ is the group generated by ι_1, ι_2. $\sigma_{\mathcal{W}} = \iota_2 \circ \iota_1$.

From now on, we will assume that $E_{\mathcal{W}}$ has genus 1 and $G_{\mathcal{W}}$ is infinite.

$E_{\mathcal{W}}$ has genus 1 $\iff$ $E_{\mathcal{W}}$ is irreducible and smooth.

$G_{\mathcal{W}}$ is infinite $\iff$ order of $\sigma = \infty$.

Note: 1) $\sigma(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$ for some $\mathbf{P} \in E_{\mathcal{W}}$ so

$G_{\mathcal{W}}$ is infinite $\iff$ $\mathbf{P}$ has infinite order $\iff$ order of $\mathbf{P} > 6$ *(Oguiso-Shioda)*.

2) $\sigma : E_{\mathcal{W}} \to E_{\mathcal{W}}$ induces an action on function on $E_{\mathcal{W}}$ by $\sigma(f(\mathbf{Q})) = f(\sigma(\mathbf{Q}))$.

The Difference Equation

The generating series satisfies

$$K_{\mathcal{W}}(x,y,t)Q_{\mathcal{W}}(x,y,t) = xy - K_{\mathcal{W}}(x,0,t)Q_{\mathcal{W}}(x,0,t) - K_{\mathcal{W}}(0,y,t)Q_{\mathcal{W}}(0,y,t) + K_{\mathcal{W}}(0,0,t)Q_{\mathcal{W}}(0,0,t).$$

Setting $K_{\mathcal{W}}(x,y,t) = 0$ we have

$$0 = -K_{\mathcal{W}}(x,0,t)Q_{\mathcal{W}}(x,0,t) - K_{\mathcal{W}}(0,y,t)Q_{\mathcal{W}}(0,y,t) + K_{\mathcal{W}}(0,0,t)Q_{\mathcal{W}}(0,0,t)$$

for $\{x|,|y| < 1\} \cap E_{\mathcal{W}}$.

$Q_{\mathcal{W}}(x,0,t)$ and $Q_{\mathcal{W}}(0,y,t)$ can be continued to multivalued meromorphic functions of $E_{\mathcal{W}}$ and that for $F = K_{\mathcal{W}}(0,y,t)Q_{\mathcal{W}}(0,y,t)$ and $b = x(\iota_1(y) - y)$we have

$$\sigma(F) - F = b$$

on $E_{\mathcal{W}}$. *F-I-M (1999), Kurkova/Raschel (2012) (unweighted), Dreyfus/Raschel (2019) (weighted)*

The Difference Equation and Differential Algebraicity

Curve: $E_{\mathcal{W}} = \overline{\{(x, y) \mid K_{\mathcal{W}}(x, y, t) = 0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$

Group: $G_{\mathcal{W}} = \langle \iota_1, \iota_2 \rangle, \sigma = \iota_2 \circ \iota_1,\ \sigma(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$; $\mathbf{P}, \mathbf{Q} \in E_{\mathcal{W}}$, $\mathbf{P}$ infinite order.

$Q_{\mathcal{W}}(0, y, t)$ can be continued to multivalued meromorphic function of $E_{\mathcal{W}}$ such that for $F = K_{\mathcal{W}}(0, y, t)Q_{\mathcal{W}}(0, y, t)$ and $b = x(\iota_1(y) - y)$ we have

$$\sigma(F) - F = b$$

on $E_{\mathcal{W}}$.

Fact: There is a derivation δ on functions on $E_{\mathcal{W}}$ such that $\delta \circ \sigma = \sigma \circ \delta$.

Prop.

$$Q_{\mathcal{W}}(x, y, t) \text{ is DA}$$
$$\Updownarrow$$
$$Q_{\mathcal{W}}(0, y, t) \text{ is } y\text{-DA}$$
$$\Updownarrow$$
$$F = K_{\mathcal{W}}(0, y, t)Q_{\mathcal{W}}(0, y, t) \text{ is DA with respect to } \delta \text{ over } E_{\mathcal{W}}$$
$$\Updownarrow$$
$$\sigma(F) - F = b \text{ has a DA solution in a } \sigma\delta\text{-extension of } \mathbb{C}(E_{\mathcal{W}}).$$

Theorems for Differential Algebraicity: Galois Theory

k a $\sigma\delta$-field, σ an automorphism, δ a derivation, $\sigma\delta = \delta\sigma$.
$k^\delta = \{c \in k \mid \delta(c) = 0\}$ alg. closed.

Prop. (Hardouin, 2006) Let $b \in k$, TFAE:

1. There exists a $\sigma\delta$ extension $k \subset K$ and $y \in K$ s.t.
   - $\sigma(y) - y = b$, and
   - y satisfies a δ-differential equation over k.
2. There exists a $\sigma\delta$ extension $k \subset K$ and $y \in K$ s.t.
   - $\sigma(y) - y = b$, and
   - $\exists g \in k,\ c_i \in k^\delta$ s.t.

$$\delta^n(y) + c_{n-1}\delta^{n-1}(y) + \ldots + c_1\delta(y) + c_0 y = g.$$

3. $\exists g \in k,\ c_i \in k^\delta$ s.t.

$$\delta^n(b) + c_{n-1}\delta^{n-1}(b) + \ldots + c_1\delta(b) + c_0 b = \sigma(g) - g.$$

Theorems for Differential Algebraicity: Certificates

Prop.(*D-H-R-S, 2018*) Let $b = x(\iota_1(y) - y) \in \mathbb{C}(E_{\mathcal{W}})$.TFAE:

1. $Q_{\mathcal{W}}(0, y, t)$ is y-DA
2. There exists an integer $n \geq 0$ and $g \in \mathbb{C}(E_{\mathcal{W}})$ such that

$$\delta^n(b) + c_{n-1}\delta^{n-1}(b) + \ldots + c_1\delta(b) + c_0 b = \sigma(g) - g$$

   for some $c_i \in \mathbb{C}$ and suitable derivation $\delta : \mathbb{C}(E_{\mathcal{W}}) \to \mathbb{C}(E_{\mathcal{W}})$.

Prop.(*H-S, 2020*) The following are equivalent:

1. $Q_{\mathcal{W}}(0, y, t)$ is y-DA
2. There exists $g \in \mathbb{C}(E_{\mathcal{W}})$ such that $b = \sigma(g) - g$

   This g is called a certificate

$$Q_{\mathcal{W}}(x, y, t) \text{ is DA}$$
$$\Updownarrow$$
$$x(\iota_1(y) - y) = \sigma(g) - g \text{ for some } g \in \mathbb{C}(E_{\mathcal{W}})$$

Theorems for DIfferential Algebraicity: Orbit Residues

Def. $E_{\mathcal{W}}$ elliptic curve, $\sigma_{\mathcal{W}}$ the addition by a non-torsion point **P**, $K = \mathbb{C}(E_{\mathcal{W}})$

- $\{u_{\mathbf{Q}} \mid \mathbf{Q} \in E_{\mathcal{W}}\}$ local param. are **coherent** if $u_{\mathbf{Q} \ominus \mathbf{P}} = \sigma(u_{\mathbf{Q}})$.
- For $g \in \mathbb{C}(E_{\mathcal{W}})$, $\mathbf{Q} \in E_{\mathcal{W}}$, write

$$g = \frac{c_{\mathbf{Q},N}}{{u_{\mathbf{Q}}}^N} + \cdots + \frac{c_{\mathbf{Q},i}}{u_{\mathbf{Q}}^i} + \cdots + \frac{c_{\mathbf{Q},1}}{u_{\mathbf{Q}}} + f$$

with f regular at **Q**. Then, the $\mathbf{i^{th}}$ **orbit residue** of g at **Q** is

$$\mathrm{ores}^i_{\mathbf{Q}}(g) = \sum_{n \in \mathbb{Z}} c^i_{\sigma^n(\mathbf{Q})}.$$

Prop. *(D-H-R-S (2018))* The following are equivalent for $b \in \mathbb{C}(E_{\mathcal{W}})$, $\iota_1(b) = -b$:

- b has a certificate.
- For all $i \in \mathbb{N}_{>0}$, $Q \in E_{\mathcal{W}}$, $\mathrm{ores}^i_Q(b) = 0$.

When this happens one can find g such that $b = \sigma(g) - g$.

To determine if $Q_{\mathcal{W}}(x, y, t)$ is DA

find the orbits of the poles of $b = x(\iota_1(y) - y)$ and their orbit residues.

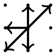
Theorems for DIfferential Algebraicity: Orbit Residues

Prop. *(D-H-R-S (2018))* The following are equivalent for $b = x(\iota_1(y) - y)$:

- b has a certificate.
- For all $i \in \mathbb{N}_{>0}, Q \in E_{\mathcal{W}}$, $\text{ores}^i_Q(b) = 0$.
- *(D-S (2020))* For two specific poles $\mathbf{N}, \mathbf{M}$ depending on $\mathcal{W}, \exists n \in \mathbb{Z}$ s.t.

$$\sigma^n(\mathbf{N}) = \mathbf{M}.$$

Ex. The weighted model



and

- $\mathbf{M} = ([1 : 0], [0 : 1])$
- $\mathbf{N} = ([-d_{0,1} : d_{1,1}], [1 : 0])$

$$Q_{\mathcal{W}}(x, y, t) \text{ is DA} \iff \mathbf{M} = \sigma^n_{\mathcal{W}}(\mathbf{N}) \text{ for some } n.$$

Theorems for DIfferential Algebraicity: Orbit Residues

In general, we have

the generating series of a weighted model $\mathcal{W}$ is differentially algebraic

$$\Updownarrow$$

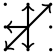
two specific poles of b lie in the same orbit.

The two poles giving this criterion depend on the relative positions of the (at most 6) poles of b and their behavior under ι_1, ι_2 and not on the weights.

The condition that these poles lie in the same orbit does depend on the weights and gives the NASC, in terms of weights, for the generating series to be DA.

How does one decide if $\exists n \in \mathbb{Z}$ s.t. $\mathbf{M} = \sigma_{\mathcal{W}}^{n}(\mathbf{N})$?

Ex. The weighted model



has differential algebraic generating series if and only

$$([1:0],[0:1]) = \sigma_{\mathcal{W}}^{1}([-d_{0,1} : d_{1,1}],[1:0]).$$
$$\Updownarrow$$
$$d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$$

How does one find 1?

Algorithms for DA: Mordell-Weil Lattices, Néron-Tate Height

Mordell-Weil-Néron Theorem. If E be an elliptic curve defined over k, a finitely generated extension of $\mathbb{Q}$ then the group $E(k)$ of k-rational points, is a finitely generated abelian group,

$$E(k) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus E(k)_{torsion}.$$

Denote $E(k)/E(k)_{torsion}$ by $\mathrm{MWL}(E)$.

Now assume E is defined over $k = \mathbb{Q}(t)$ and that E does not descend to $\mathbb{Q}$.

There is a $\mathbb{Q}$-valued symm. bilinear form

$$\langle *, * \rangle : E(k) \times E(k) \to \mathbb{Q}$$

called the Néron-Tate Pairing and the quadratic form

$$\hat{h}(\mathbf{Q}) = \langle \mathbf{Q}, \mathbf{Q} \rangle$$

is called the Néron-Tate Height.

Algorithms for DA: Mordell-Weil Lattices, Néron-Tate Height

$$k = \mathbb{Q}(t)$$

(Oguiso-Shioda): As groups, there are 26 possibilites for for $E(k)$. The order of any element is at most 6 or infinite.

Properties of $\hat{h}$:

- If $\mathbf{N}$ is a torison point, then $\hat{h}(\mathbf{N}) = 0$.

  $\sigma_{\mathcal{W}} : \mathbf{Q} \mapsto \mathbf{Q} \oplus \mathbf{P}$ has finite order iff $\hat{h}(\mathbf{P}) = 0$
- If $\mathbf{M} = n\mathbf{N}$, then $\hat{h}(\mathbf{M}) = n^2 \hat{h}(\mathbf{N})$.
- Can reduce finding n s.t. $\mathbf{M} = \sigma_{\mathcal{W}}^n(\mathbf{N})$ to fnding n s.t. $\mathbf{M} = n\sigma_{\mathcal{W}}(\mathbf{N})$.
- $\hat{h}(\mathbf{N})$ is computable. *For the points we consider, this depends on the configuration of the 8 points common to all curves in the family $K(x, y, t) = 0$ (base points), not on the weights.*

Algorithms for DA

Fix a rational step set $\mathcal{W}$ such that the curve is an elliptic curve $E_{\mathcal{W}}$ and the $G_{\mathcal{W}} = \langle \iota_1, \iota_2 \rangle$ is infinite. Let $\sigma_{\mathcal{W}}(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$.

The generating series is DA

$$\Updownarrow$$

$b = x(\iota_1(y) - y)$ has a certificate.

$$\Updownarrow$$

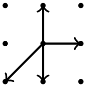
There are two poles of b, $\mathbf{M}, \mathbf{N} \in E_{\mathcal{W}}(\mathbb{Q}(t))$, such that $\sigma_{\mathcal{W}}^n(\mathbf{N}) = \mathbf{M}$, $n \in \mathbb{Z}$
$\mathbf{M}, \mathbf{N}$ depend only on $\mathcal{D}$, not on the weights.

$$\Updownarrow$$

Determine if $\exists n \in \mathbb{Z}$ s.t. $\hat{h}(\mathbf{M}) = n^2 \hat{h}(\sigma_{\mathcal{W}}(\mathbf{N}))$.
If no, the generating series is not DA.
If yes, the condition $\sigma_{\mathcal{W}}^n(\mathbf{N}) = \mathbf{M}$ yields polynomial conditions on the weights giving DA.

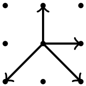
Ex.

DA gen. series $\Leftrightarrow ([1:0],[0:1]) = \sigma_{\mathcal{W}}^{1}([-d_{0,1} : d_{1,1}],[1:0]) \Leftrightarrow d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$



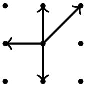
$w_{\mathrm{IIB.1}}$

All



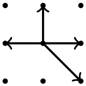
$w_{\mathrm{IIB.2}}$

All



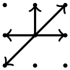
$w_{\mathrm{IIC.1}}$

All



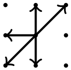
$w_{\mathrm{IIB.3}}$

All



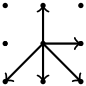
$w_{\mathrm{IIC.4}}$

$d_{-1,-1}d_{1,1} - d_{1,0}d_{-1,0} = 0$



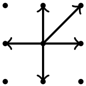
$w_{\mathrm{IIC.2}}$

$d_{0,1}d_{0,-1} - d_{1,1}d_{-1,-1} = 0$



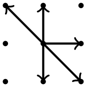
$w_{\mathrm{IIB.6}}$

All



$w_{\mathrm{IIC.5}}$

All



$w_{\mathrm{IIB.7}}$

$d_{-1,1}d_{1,-1} - d_{0,-1}d_{0,1} = 0$