Moving frames for partial difference equations

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For ordinary difference equations, see
EL Mansfield, A Rojo-Echeburúa, PE Hydon & L Peng
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The total space

For real (topologically trivial) partial difference equations (PΔEs) on $\mathbb{Z}^m$, the total space is $\mathcal{T} = \mathbb{Z}^m \times \mathbb{R}^q$, with coordinates

$$\mathbf{n} = (n^1, \ldots, n^m) \quad \text{(ordered independent variables)}$$
$$\mathbf{u} = (u^1, \ldots, u^q) \quad \text{(dependent variables)}$$

Dependent variables coordinatize the continuous fibres $\mathcal{T}_{\mathbf{n}} = \mathbb{R}^q$.

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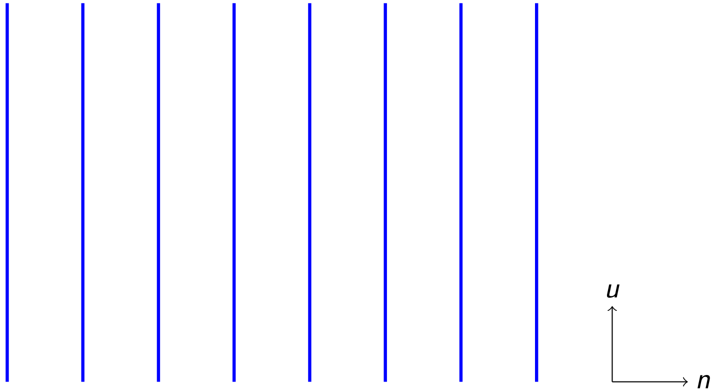
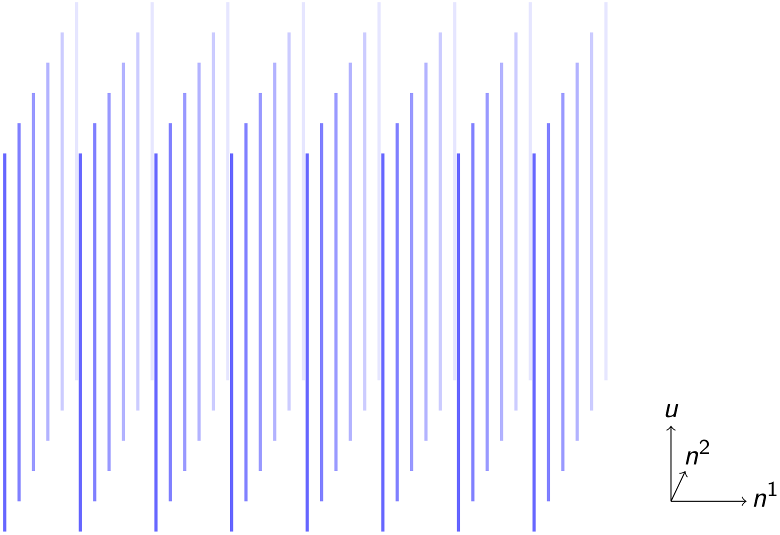
Note: everything generalizes to equations on lattice varieties.

The total space: $\mathbb{Z} \times \mathbb{R}$



The total space: $\mathbb{Z}^2 \times \mathbb{R}$



The total space $\mathcal{T}$ is invariant under every translation

$$\mathrm{T}_{\mathbf{J}} : \mathcal{T}_{\mathbf{n}} \longrightarrow \mathcal{T}_{\mathbf{n}+\mathbf{J}}, \qquad \mathrm{T}_{\mathbf{J}} : (\mathbf{n}, \mathbf{u}) \longmapsto (\mathbf{n}+\mathbf{J}, \mathbf{u}), \qquad \mathbf{J} \in \mathbb{Z}^m.$$

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To represent $\mathcal{T}$ as a connected space over a given $\mathbf{n}$, prolong $\mathcal{T}_{\mathbf{n}}$ to $P(\mathcal{T}_{\mathbf{n}})$, the infinite product space with coordinates

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Each $\mathbf{n}$ has a prolongation space $P(\mathcal{T}_{\mathbf{n}})$. Composing pullbacks gives

$$u^{\alpha}_{\mathbf{J}+\mathbf{K}} = \mathrm{T}^{*}_{\mathbf{K}}(u^{\alpha}_{\mathbf{J}}),$$

which relates the coordinates on $P(\mathcal{T}_{\mathbf{n}})$ and $P(\mathcal{T}_{\mathbf{n}+\mathbf{K}})$.

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$$\mathrm{S}_{\mathbf{K}} : {}^{f}P(\mathcal{T}_{\mathbf{n}}) \longrightarrow {}^{f}P(\mathcal{T}_{\mathbf{n}}),$$

$$\mathrm{S}_{\mathbf{K}} : f(\mathbf{n}, \ldots, u^{\alpha}_{\mathbf{J}}, \ldots) \longmapsto f(\mathbf{n}+\mathbf{K}, \ldots, u^{\alpha}_{\mathbf{J}+\mathbf{K}}, \ldots).$$

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From here on, we work with the connected prolongation space $P(\mathcal{T}_{\mathbf{n}})$ (for any fixed $\mathbf{n}$), rather than the disconnected total space. All functions are assumed to be in ${}^{f}P(\mathcal{T}_{\mathbf{n}})$.

Difference divergences

Let $\mathrm{S}_i = \mathrm{S}_{\mathbf{1}_i}$ and $\mathrm{id} = \mathrm{S}_{\mathbf{0}}$. Then the forward difference in the n^i-direction is represented by

$$\mathrm{D}_{n^i} := \mathrm{S}_i - \mathrm{id}.$$

A (difference) *divergence* is a function $\mathcal{C}$ of the form $\mathcal{C} = \mathrm{D}_{n^i}\mathrm{F}^i$.

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Lemma Every expression $(\mathrm{S}_{\mathbf{J}} - \mathrm{id})F$ is a divergence.

Theorem A function $\mathcal{C}$ is a divergence if and only if

$$\mathbf{E}_{u^\alpha}(\mathcal{C}) := \mathrm{S}_{-\mathbf{J}}\left(\frac{\partial \mathcal{C}}{\partial u^\alpha_{\mathbf{J}}}\right) = 0, \qquad \alpha = 1, \ldots, m.$$

Here $\mathbf{E}_{u^\alpha}$ is the Euler–Lagrange operator for u^α.

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where each $f^{\mathbf{J}}$ is a function.

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The formal adjoint of a given difference operator $\mathcal{H}$ is the unique difference operator $\mathcal{H}^{\dagger}$ such that

$$F_1(\mathcal{H}F_2) - \left(\mathcal{H}^{\dagger}F_1\right)F_2$$

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A *conservation law* of a given system of PΔEs is a divergence that is zero on all solutions.

Noether's Theorem

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The generalized symmetry with characteristic $\mathbf{Q}(\mathbf{n}, [\mathbf{u}])$,

$$\mathbf{v} = (\mathrm{S}_\mathbf{J} Q^\alpha) \frac{\partial}{\partial u^\alpha_\mathbf{J}},$$

is a variational symmetry if $\mathbf{v}(\mathrm{L}) = \mathrm{D}_{n^i} \mathrm{F}^i$, for some F^i.

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is a variational symmetry if $\mathbf{v}(\mathrm{L}) = \mathrm{D}_{n^i}\mathrm{F}^i$, for some F^i. Then

$$\mathrm{D}_{n^i}\mathrm{F}^i - (\mathrm{S}_{\mathbf{J}} - \mathrm{id})\left\{ Q^\alpha \mathrm{S}_{-\mathbf{J}} \left(\frac{\partial \mathrm{L}}{\partial u^\alpha_{\mathbf{J}}} \right) \right\} = \mathrm{D}_{n^i}\mathrm{F}^i - \mathbf{v}(\mathrm{L}) + Q^\alpha \mathbf{E}_{u^\alpha}(\mathrm{L}),$$

so every variational symmetry yields a conservation law for the E–L equations. The converse is also true.

Difference moving frames

Now consider a Lie group G of point transformations whose (left) action on $P(\mathcal{T}_{\mathbf{n}})$ is free and regular.

Each characteristic $\mathbf{Q}(\mathbf{n},\mathbf{u})$ gives a one-parameter Lie subgroup,

$$g_\varepsilon : P(\mathcal{T}_{\mathbf{n}}) \longrightarrow P(\mathcal{T}_{\mathbf{n}}), \qquad g_\varepsilon \cdot u^\alpha_{\mathbf{J}} = \exp(\varepsilon \mathbf{v}) u^\alpha_{\mathbf{J}}.$$

Similarly, the action of each $g \in G$ on u^α prolongs to $g \cdot u^\alpha_{\mathbf{J}}$.

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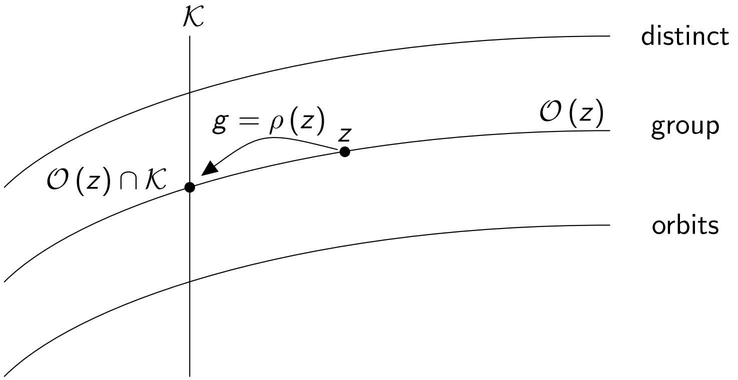
If G is R-dimensional, choose a (local) cross-section $\mathcal{K}$ transverse to the group orbits, defined by

$$\psi_r(z) = 0, \qquad r = 1, \ldots, R.$$

Where possible, we choose z to be a set of R coordinates from $[\mathbf{u}]$ (typically, values of $\mathbf{u}$ at $\mathbf{n}$ and nearby points).



Moving frame defined by a cross-section

A *difference moving frame* is an equivariant map $\rho : P(\mathcal{T}_{\mathbf{n}}) \rightarrow G$, which is obtained by solving the normalization equations,

$$\psi_r(g \cdot z) = 0, \qquad r = 1, \ldots, R,$$

for the group parameters, giving $g = \rho(z)$.

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A difference moving frame represents a discrete moving frame (on a finite prolongation of $\mathcal{T}$) that is invariant under all $\mathrm{T}_{\mathbf{K}}$.
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A function f is G-invariant if $f(\mathbf{n}, [g \cdot \mathbf{u}]) = f(\mathbf{n}, [\mathbf{u}])$, for all $g \in G$.

The *invariantization*, ι, is defined by $\iota(f(\mathbf{n}, [\mathbf{u}])) = f(\mathbf{n}, [\rho(z) \cdot \mathbf{u}])$.

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The set of all G-invariants is generated by the invariants $\iota(u^{\alpha}_{\mathbf{0}})$ and the Maurer–Cartan invariants $(\mathrm{S}_i \rho(z))(\rho(z))^{-1}$.

Example The Lagrangian

$$\mathrm{L} = \frac{1}{2} \ln \left| \frac{(u_{2,0} - u_{1,1})(u_{1,-1} - u_{0,0})}{(u_{2,0} - u_{1,-1})(u_{1,1} - u_{0,0})} \right|$$

yields a Toda-type Euler–Lagrange equation,

$$\mathbf{E}_u \mathrm{L} = \frac{1}{u_{1,1} - u_{0,0}} - \frac{1}{u_{-1,1} - u_{0,0}} - \frac{1}{u_{1,-1} - u_{0,0}} + \frac{1}{u_{-1,-1} - u_{0,0}} = 0.$$

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The Lie group of variational point symmetries is generated by

$$Q_1 = 1,\ \ Q_2 = u_{0,0},\ \ Q_3 = u_{0,0}^2,\ \ (Q_4, Q_5, Q_6) = (-1)^{n^1+n^2}(Q_1, Q_2, Q_3).$$

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We use the subgroup generated by Q_1 and Q_2:

$$g \cdot u_{i,j} = a u_{i,j} + b, \qquad a \in \mathbb{R}^+,\ \ b \in \mathbb{R}.$$

For $u_{1,1} > u_{0,0}$, a useful normalization is $g \cdot u_{0,0} = 0, \; g \cdot u_{1,1} = 1$. Then the frame ρ is defined by

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The Maurer-Cartan invariants yield generating invariants,

$$\kappa = \iota\left(u_{1,-1}\right), \qquad \lambda = \iota\left(u_{2,0}\right).$$

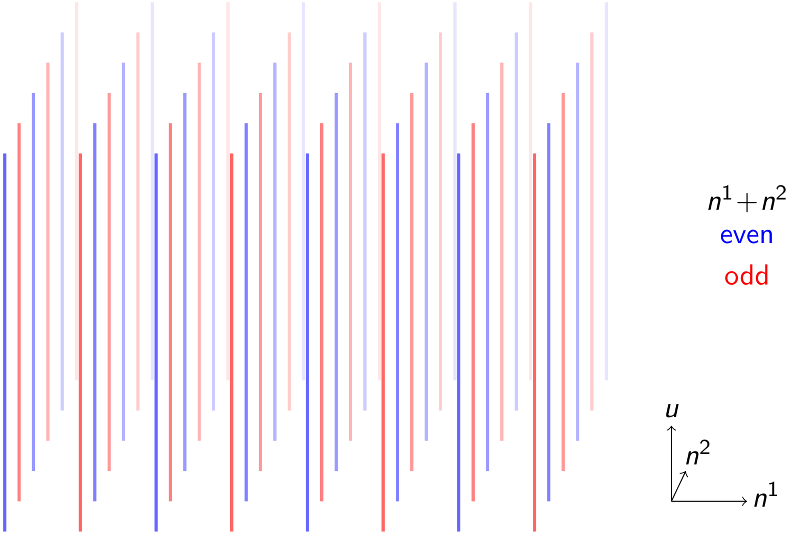
Then

$$\iota(u_{i,j}) = \frac{u_{i,j} - u_{0,0}}{u_{1,1} - u_{0,0}},$$

which leads to the invariantized Lagrangian,

$$L := \iota(\mathrm{L}) = \frac{1}{2} \ln \left| \frac{(\lambda - 1)\,\kappa}{\lambda - \kappa} \right|.$$

Partitioned total space for the Toda-type equation



Invariant Euler–Lagrange equations

Suppose that the generating invariants are κ^β and that the Lagrangian is invariant under the group action. Define

$$L(\mathbf{n}, [\boldsymbol{\kappa}]) := \iota(\mathrm{L}(\mathbf{n}, [\mathbf{u}])) = \mathrm{L}(\mathbf{n}, [\mathbf{u}]).$$

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Introduce an invariant dummy variable $t \in \mathbb{R}$ that parametrizes an arbitrary smooth path in $P(\mathcal{T}_\mathbf{n})$. Then on this path,

$$\mathrm{L}' = \frac{\partial \mathrm{L}}{\partial u^\alpha_\mathbf{J}}(u^\alpha_\mathbf{J})' = \mathbf{E}_{u^\alpha}(\mathrm{L})(u^\alpha_\mathbf{0})' + \mathrm{D}_{n^i}\mathrm{F}^i(\mathbf{n},[\mathbf{u}],[\mathbf{u}']).$$

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Therefore,

$$L' = \iota\left\{\mathbf{E}_{u^\alpha}(\mathrm{L})\right\}\sigma^\alpha + \iota\left\{\mathrm{D}_{n^i}\mathrm{F}^i(\mathbf{n},[\mathbf{u}],[\mathbf{u}'])\right\}.$$

where $\sigma^\alpha = \iota\{(u^\alpha_{\mathbf{0}})'\}$. The rightmost term is a divergence!

Similarly,

$$L' = \frac{\partial L}{\partial \kappa^\beta_{\mathbf{J}}} (\kappa^\beta_{\mathbf{J}})' = \mathbf{E}_{\kappa^\beta}(L)(\kappa^\beta)' + \mathrm{D}_{n^i} \left\{ F^i_\beta(\mathbf{n}, [\boldsymbol{\kappa}])(\kappa^\beta)' \right\},$$

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involve invariant difference operators $\mathcal{H}^\beta_\alpha$ that are found by writing κ^β in terms of $[\mathbf{u}]$. Therefore,

$$L' = (\mathcal{H}^\beta_\alpha)^\dagger\left\{\mathbf{E}_{\kappa^\beta}(L)\right\}\sigma^\alpha + \mathrm{D}_{n^i}\left\{F^i_\beta(\mathbf{n},[\boldsymbol{\kappa}])(\kappa^\beta)' + H^i_\alpha(\mathbf{n},[\boldsymbol{\kappa}])\sigma^\alpha\right\},$$

for difference operators H^i_α. So the invariantized E–L equations are

$$\iota\left(\mathbf{E}_{u^\alpha}(\mathrm{L})\right) = (\mathcal{H}^\beta_\alpha)^\dagger\left\{\mathbf{E}_{\kappa^\beta}(L)\right\}.$$

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to

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On the path parametrized by t,

$$\sigma^\alpha = \iota(Q_s^\alpha) a_r^s(\rho(z)),$$

where a_r^s are components of the Adjoint matrix. This gives an invariant form of Noether's Theorem!

Questions?