Pseudo-difference operators and discrete W_n algebras

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Discrete Geometric Realizations of Integrable Systems

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Authors: Bobenko, Calini, Doliwa& Santini, Fukujioka& Kurose, Hoffman, Mansfield, Marí Beffa, Inoguchi-Kajiwara & Matsuura, Wang, Surisé and many more.

Joint work with A. Izosimov, "*What is a Lattice W_m-Algebra?*" (in preparation), based on work with JP Wang and A. Calini.

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Under natural conditions $\{K_n\}_{n=1}^N$ will define coordinates in the moduli space of polygons. (Mansfield, MB, Wang 2013).

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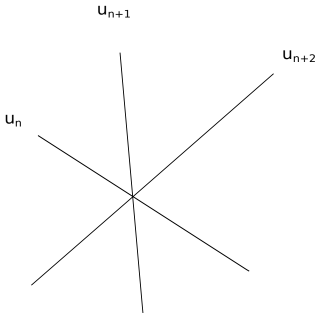

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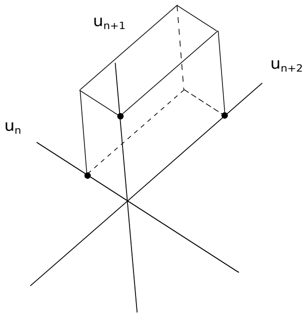

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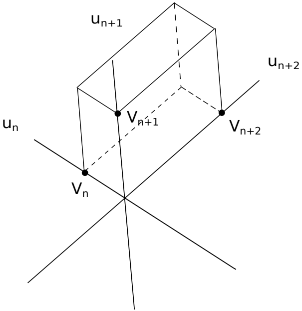

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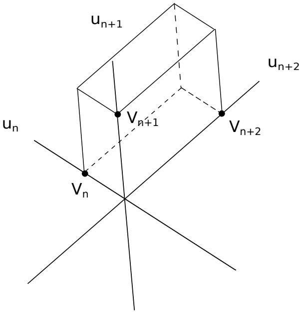

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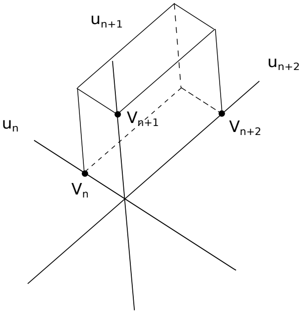

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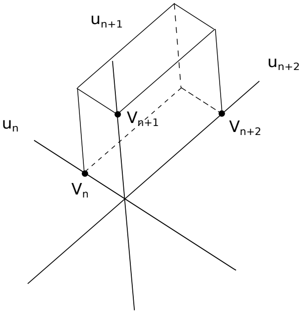

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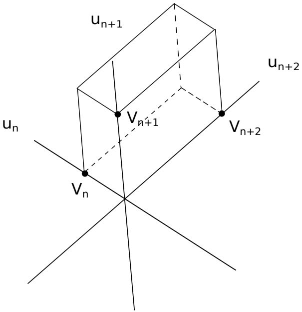

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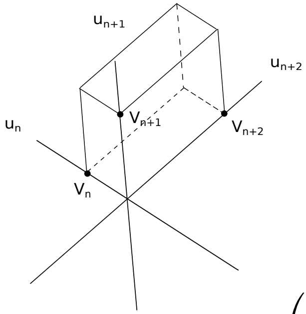

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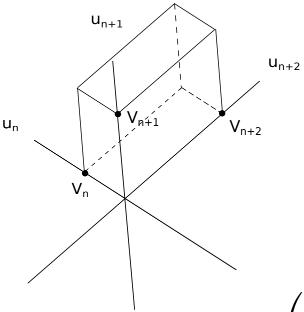

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$\{a_n^i\}$ functionally generate all other invariants (Schwartz, Ovsienko and Tabachnikov, 2010).

In general

Theorem

(MB 2014) Assume $M = G/H$. The moduli space of non degenerate twisted polygons in M^N can be identified with an open subset of the quotient G^N/H^N, where H^N acts on G^N via the right gauge action

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They were classified by Semenov-Tian-Shansky in "Dressing transformations and Poisson Group actions", (1985). We will describe the main such bracket.

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The twisted Poisson bracket defines a Hamiltonian structure for which gauge action is a Poisson map.

A discrete geometric Poisson bracket

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$$(V_n)_t = X_n^f.$$

The projective case

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(which is not a Poisson bracket), also reduces to G^N/H^N to produce a second Hamiltonian structure for the same integrable system.

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$$\langle L, \hat{L} \rangle = \mathrm{Tr}(L\hat{L}).$$

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$$[Rx, Ry] - R[Rx, y] - R[x, Ry] = -[x, y], \text{ for any } x, y \in PDO_m^N.$$

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and also to the case where $a_j^0 = (-1)^{m-1}$ for all j.

Given $L = (-1)^{m-1} + a^1\mathcal{T} + \cdots + a^{m-1}\mathcal{T}^{m-1} - \mathcal{T}^m$, let $\{V_n\}$ be a twisted bi-infinite sequence defined by its kernel $LV = 0$

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$$L_n V_n = ((-1)^{m-1} + a_n^1\mathcal{T} + \cdots + a_n^{m-1}\mathcal{T}^{m-1} - \mathcal{T}^m)V_n = 0 \text{ for all } n,$$

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$$(LV)_t = L_t V + LV_t = \left(R(LQ^f)L - LR(Q^f L)\right) V + LY^f = -LR(Q^f L)V + LY^f = 0$$

That is, $Y^f = R(Q^f L)V$ up to the kernel of L. Y^f is unique if preserving $a^0 = (-1)^{m-1}$.
Let $(V_n)_t = X_n^f$, where X_n^f is the geometric realization for the bracket in the first half.

Given $L = (-1)^{m-1} + a^1\mathcal{T} + \cdots + a^{m-1}\mathcal{T}^{m-1} - \mathcal{T}^m$, let $\{V_n\}$ be a twisted bi-infinite sequence defined by its kernel $LV = 0$

$$L_n V_n = ((-1)^{m-1} + a_n^1\mathcal{T} + \cdots + a_n^{m-1}\mathcal{T}^{m-1} - \mathcal{T}^m)V_n = 0 \text{ for all } n,$$

unique up to the diagonal action of the group, with a_n^k projective generating invariants.

Let Y^f be a geometric realization, a polygonal vector field inducing a f-Hamiltonian evolution on invariants

$$(LV)_t = L_t V + LV_t = \left(R(LQ^f)L - LR(Q^fL)\right) V + LY^f = -LR(Q^fL)V + LY^f = 0$$

That is, $Y^f = R(Q^fL)V$ up to the kernel of L. Y^f is unique if preserving $a^0 = (-1)^{m-1}$.

Let $(V_n)_t = X_n^f$, where X_n^f is the geometric realization for the bracket in the first half.

Theorem

(Isozimov, MB 2021) If f is a Hamiltonian function on the moduli space, then

$$X^f = Y^f$$

and so both Poisson brackets are identical.

MERCI!   THANKS!