Algebraic Moving Frame and Beyond

Sections and the computation of rational invariants

Evelyne Hubert

Inria & Université Côte d'Azur

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Based on joint works with either I. Kogan, or G. Labahn, or P. Görlach and invaluable discussions with E. Mansfield & P. Olver

Sections and the computation rational invariants for applications

1. Construction of rational invariants : a general algorithm

2. Scalings and parameter reduction in mathematical models for biology without fractional powers

3. Orthogonal invariants of ternary quartics and neuro-imaging

Rational action $\star$ of an affine algebraic group $\mathcal{G}$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Group action:

$$\begin{array}{rrcl} \star : & \mathcal{G} \times \mathbb{K}^n & \rightarrow & \mathbb{K}^n \\ & (\lambda, z) & \mapsto & \lambda \star z \end{array} \qquad \text{s.t} \qquad \begin{array}{rcl} 1 \star z & = & z \\ (\lambda \cdot \mu) \star z & = & \lambda \star (\mu \star z) \end{array}$$

$\mathcal{G} \subset \mathbb{K}^l$ an algebraic variety

$G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_l]$ its ideal

Rational action of $\mathcal{G}$ on $\mathbb{K}^n$

$$\lambda \star z = \left(\frac{p_1(\lambda, z)}{q(\lambda, z)}, \ldots, \frac{p_n(\lambda, z)}{q(\lambda, z)} \right)$$

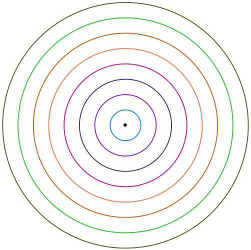
$q, p_1, \ldots, p_n \in \mathbb{K}[\lambda_1, \ldots, \lambda_l, z_1, \ldots, z_n]$

Orbit $\mathcal{O}_z$ of $z \in \mathbb{K}^n$: the image of $\mathcal{G}$ under $\lambda \mapsto \lambda \star z$

Linear actions in the plane

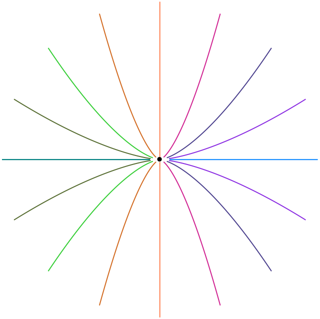
$$\mathcal{G} = \mathrm{SO}_2, \quad G = \left(\lambda^2 + \mu^2 - 1\right)$$

$$\lambda \star \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$



$$\mathcal{G} = \mathbb{K}^*, \quad G = (\lambda\,\mu - 1)$$

$$\lambda \star \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda^2\, z_1 \\ \lambda^3\, z_2 \end{pmatrix}$$



Rational invariants

$\star : \mathcal{G} \times \mathcal{Z} \rightarrow \mathcal{Z}$ $\qquad$ $\mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$

Rational invariant: $f \in \mathbb{K}(z_1, \ldots, z_n)$ s.t. $f(\lambda \star z) = f(z), \ \forall \lambda \in \mathcal{G}$

Field of rational invariants: $\mathbb{K}(z)^{\mathcal{G}}$ $\qquad$ finitely generated

THM: $\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_k) \ \Leftrightarrow \ \{r_1, \ldots, r_k\}$ separating $\qquad$ [Rosenlicht 56]

Separating: $r_1(z) = r_1(z'), \ldots, r_k(z) = r_k(z') \ \Leftrightarrow \ z' \in \mathcal{O}_z$ $\quad$ for $z, z' \in \mathcal{Z} \setminus \mathcal{W}$

Section

Section of degree e :

An irreducible variety $\mathcal{P}$ that intersects generic orbits in e points.

f.i. a generic affine space of complementary dimension to the orbit

Linear actions in the plane

$\mathcal{G} = \mathrm{SO}_2, \quad G = (\lambda^2 + \mu^2 - 1)$

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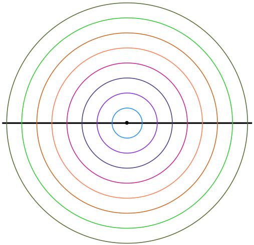

$$Q = \{Y, X^2 - (x^2 + y^2)\}$$

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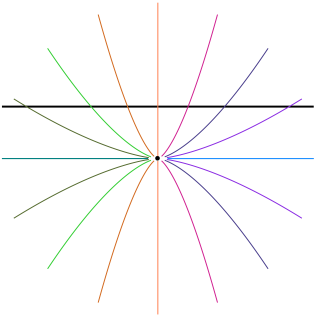

$$Q = \left\{ Y - 1, X^3 - \frac{x^3}{y^2} \right\}$$

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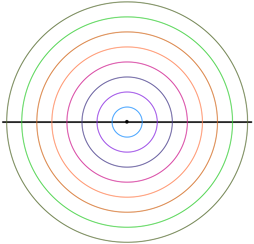

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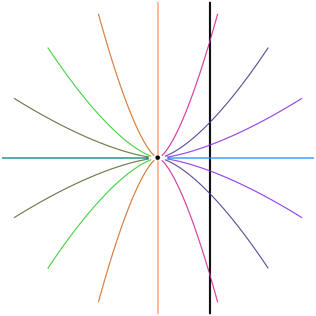

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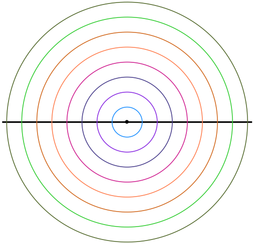

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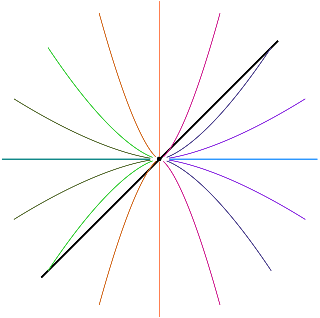

$$Q = \left\{ Y - \frac{x^3}{y^2}, X - \frac{x^3}{y^2} \right\}$$

Section & intersection ideal

Section of degree e :

An irreducible variety $\mathcal{P}$ that intersects generic orbits in e points.

f.i. a generic affine space of complementary dimension to the orbit

Intersection ideal: $I \subset \mathbb{K}(z_1, \ldots, z_n)[Z_1, \ldots, Z_n]$

Under specialization $z_i \mapsto \bar{z}_i \in \mathbb{K}$ $I_{\bar{z}} \subset \mathbb{K}[Z]$ is the ideal of $\mathcal{O}_{\bar{z}} \cap \mathcal{P}$

Prp: $I_{\lambda \star \bar{z}} = I_{\bar{z}}$

$\rightsquigarrow$ A canonical representation of I has coefficients in $\mathbb{K}(z)^{\mathcal{G}}$

$\rightsquigarrow$ These coefficients generate $\mathbb{K}(z)^{\mathcal{G}}$ by the separation property

f.i. [Rosenlich 56] considered the Chow form of I

Intersection ideal as an elimination ideal

$$I = (G + (Z - \lambda \star z) + P) \,\cap\, \mathbb{K}(z)[Z]$$

Example :

$$G = (\lambda^2 + \mu^2 - 1), \quad (Z - \lambda \star z) = (X - \lambda x + \mu y,\ Y - \mu x - \lambda y), \quad P = (Y)$$

- P a prime ideal in $\mathbb{K}[Z]$, $\mathcal{P} = \mathcal{V}(P)$ an irreducible variety of complementary dimension to the generic orbits

Intersection ideal as an elimination ideal

$$I = (G + (Z - \lambda \star z) + P) : q^{\infty} \;\cap\; \mathbb{K}(z)[Z]$$

Example :

$$G = (\lambda^2 + \mu^2 - 1), \quad (Z - \lambda \star z) = (X - \lambda x + \mu y,\; Y - \mu x - \lambda y), \quad P = (Y)$$

- P a prime ideal in $\mathbb{K}[Z]$, $\mathcal{P} = \mathcal{V}(P)$ an irreducible variety of complementary dimension to the generic orbits
- When $\lambda \star z = \left(\frac{p_1(\lambda, z)}{q(\lambda, z)}, \ldots, \frac{p_n(\lambda, z)}{q(\lambda, z)} \right)$

$$(Z - \lambda \star z) = (q(\lambda, z)\, Z_1 - p_1(\lambda, z), \ldots, q(\lambda, z)\, Z_n - p_n(\lambda, z))$$

Invariants from the reduced Gröbner basis

[Hubert Kogan JSC 2007]

$$I = (P + (Z - \lambda \star z) + G) : q^\infty \cap \mathbb{K}(z)[Z]$$

Q reduced Gröbner basis of I

$\{r_1, \ldots, r_k\}$ its coefficients

Thm : $\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_k)$

Pf: Rewriting $\dfrac{p}{q} \in \mathbb{K}(z)^{\mathcal{G}}$

$y_1, \ldots, y_k$ a new indeterminates

$Q_y := Q(r_i \leftarrow y_i)$

$p(Z) \longrightarrow^*_{Q_y} \sum_\alpha a_\alpha(y)\, Z^\alpha$

$q(Z) \longrightarrow^*_{Q_y} \sum_\alpha b_\alpha(y)\, Z^\alpha$

$$\frac{p(z)}{q(z)} = \frac{a_\alpha(r)}{b_\alpha(r)}$$

Note : we do not need the action to be (locally) free.

Retrieving the classical invariants of SL_2 actions

- The action of $SL_2(\mathbb{C})$ on *forms* $z_0x^2 + z_1xy + z_2y^2$ of degree 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

$$I = \left(\underbrace{Z_0 - 1,\ Z_1}_{P}, Z_2 + \frac{1}{4}\left(z_1^2 - 4\, z_0 z_2\right) \right)$$

- Projective action of $SL_2(\mathbb{R})$ on quadrapules of $\mathbb{R}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} \frac{a\,z_0+b}{c\,z_0+d} & \frac{a\,z_1+b}{c\,z_1+d} & \frac{a\,z_2+b}{c\,z_2+d} & \frac{a\,z_3+b}{c\,z_3+d} \end{pmatrix}$$

$$I = \left(\underbrace{Z_0^{-1}, Z_1, Z_2 - 1}_{P}, Z_3 - \frac{(z_0-z_2)(z_1-z_3)}{(z_0-z_3)(z_1-z_2)} \right)$$

Matrix similarity

[J. Symb. Comp 2019]

Action: $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$$\begin{array}{ccc} \mathrm{SL}_n(\mathbb{K}) \times \mathrm{M}_n(\mathbb{K}) & \rightarrow & \mathrm{M}_n(\mathbb{K}) \\ (P, M) & \mapsto & P^{-1}\, M\, P \end{array}$$

Section: Companion matrices are normal forms for matrices M s.t. $\operatorname{discr} \chi(M) \neq 0$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \chi_0 \\ 1 & \cdot & \cdot & \chi_1 \\ \cdot & \ddots & \cdot & \vdots \\ \cdot & \cdot & 1 & \chi_{n-1} \end{pmatrix}$$

Invariants: The coefficients of the characteristic polynomial

$$\chi_0, \ldots, \chi_{n-1} \;: \mathrm{M}_n(\mathbb{K}) \rightarrow \mathbb{K}$$

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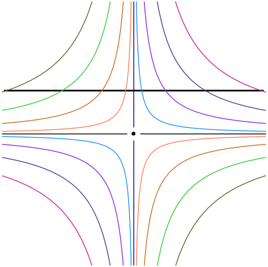
Scaling in the plane : rational sections

$A = \begin{bmatrix} a & b \end{bmatrix}$

$$\begin{array}{rccc} \star : & \mathbb{K}^* \times \mathbb{K}^2 & \to & \mathbb{K}^2 \\ & (\lambda, (x, y)) & \mapsto & (\lambda^a x, \lambda^b y) \end{array}$$

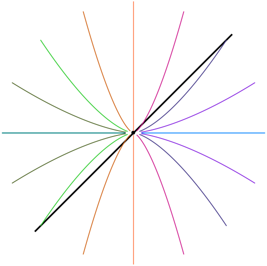
$A = \begin{bmatrix} 1 & -1 \end{bmatrix}$



$Y = 1$

$Q = \{Y - 1, X - xy\}$

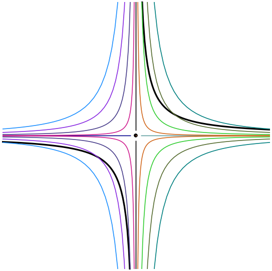
$A = \begin{bmatrix} 2 & 3 \end{bmatrix}$



$XY^{-1} = 1$

$Q = \left\{Y - \frac{x^3}{y^2}, X - \frac{x^3}{y^2}\right\}$

$A = \begin{bmatrix} 2 & -3 \end{bmatrix}$



$X^2 Y = 1$

$Q = \left\{Y - (x^3 y^2)^2, X - (x^3 y^2)^{-1}\right\}$

Scalings in the plane : the invariants

$A = \begin{bmatrix} a & b \end{bmatrix}$

$$\begin{array}{rccc} \star : & \mathbb{K}^* \times \mathbb{K}^2 & \to & \mathbb{K}^2 \\ & (\lambda, [x\ ,\ y]) & \mapsto & \left[\lambda^a x\ ,\ \lambda^b y\right] \end{array}$$

Invariant: $g = x^c y^d$ such that $(\lambda^a x)^c \left(\lambda^b y\right)^d = x^c y^d$

i.e. $\lambda^{ac+bd} x^c y^d = x^c y^d$

i.e. $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = 0$

for instance $c = -b$ and $d = a$.

Scalings in the plane : the invariants

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Generating Invariant: $g = \dfrac{y^c}{x^d}$ with $a = h\,c$ and $b = h\,d$

$h = \gcd(a, b)$

Scalings in the plane : the invariants

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Generating Invariant: $g = \dfrac{y^c}{x^d}$ with $a = h\,c$ and $b = h\,d$

$h = \gcd(a, b)$

Bezout identity : $h = \alpha\, a + \beta\, b$

$x^\alpha y^\beta = 1$ is a rational section

Moving frame : $\lambda^h = x^{-\alpha} y^{-\beta}$

Scalings in the plane : invariants and rational sections

$A = \begin{bmatrix} a & b \end{bmatrix}$

$$\star : \begin{array}{ccc} \mathbb{K}^* \times \mathbb{K}^2 & \to & \mathbb{K}^2 \\ (\lambda, [x\ ,\ y]) & \mapsto & \left[\lambda^a x\ ,\ \lambda^b y\right] \end{array}$$

Generating Invariant: $g = \dfrac{y^c}{x^d}$ with $a = h\,c$ and $b = h\,d$

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Bezout identity : $h = \alpha\, a + \beta\, b$ $x^\alpha y^\beta = 1$ is a rational section

Hermite normal form

$$\underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} \alpha & -d \\ \beta & c \end{bmatrix}}_{\text{multiplier}} = \underbrace{\begin{bmatrix} h & 0 \end{bmatrix}}_{\text{Hermite form}} .$$

Hermite Form

$H \in \mathbb{Z}^{r\times n}$, rank $r < n$ in (column) Hermite normal form if

$$H = \begin{bmatrix} 7 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \\ & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 2 & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Zero elements in right columns.
Upper triangular in left columns with nonnegative entries.
Diagonal entries in left columns largest in each row.

With integer column operation, we can always transform any integer matrix A to a column Hermite form.

Scalings : their invariants and rewrite rules

$A \in \mathbb{Z}^{r \times n}$ of rank $r \leq n$

$$\exists V \in \mathbb{Z}^{n \times n}, \quad A\,V = \left[\begin{array}{cc} H & 0 \end{array}\right], \qquad \det V = \pm 1$$

Scalings : their invariants and rewrite rules

$A \in \mathbb{Z}^{r \times n}$ of rank $r \leq n$

$$\exists V \in \mathbb{Z}^{n \times n}, \quad A \begin{bmatrix} V_{\mathfrak{i}} & V_{\mathfrak{n}} \end{bmatrix} = \begin{bmatrix} H & 0 \end{bmatrix}, \quad \det V = \pm 1$$

The columns of $V_{\mathfrak{n}}$ form a $\mathbb{Z}$-basis for $\ker A \cap \mathbb{Z}^n$

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$A \in \mathbb{Z}^{r\times n}$ defines a scaling

$$\begin{array}{rcl} (\mathbb{K}^*)^r \times \mathbb{K}^n & \rightarrow & \mathbb{K}^n \\ (\lambda\ ,\ z) & \mapsto & \begin{bmatrix}\lambda_1^{a_{11}}\dots\lambda_r^{a_{r1}}z_1 & \dots & \lambda_1^{a_{1n}}\dots\lambda_r^{a_{rn}}z_n\end{bmatrix} \end{array}$$

- the column of $V_{\mathfrak{n}}$ are the exponents of monomials $\begin{bmatrix} g_1 & \dots & g_{n-r} \end{bmatrix}$ that form a minimal generating set invariants
- the column of $V_{\mathfrak{i}}$ are the exponents of r monomials that define a rational section
- the bottom rows of $V^{-1} = \begin{bmatrix} W_{\mathfrak{u}} \\ W_{\mathfrak{d}} \end{bmatrix}$ are the exponents of $\mathfrak{n}$ monomials providing the rewrite rules $z \rightarrow g^{W_{\mathfrak{d}}}$

Parameter reduction

Prey-predator model [Murray, Mathematical Biology (2002)]

$$\begin{cases} \dot{n} &= \left(\left(1-\frac{n}{k_1}\right) r - k_2 \frac{p}{n+e}\right) n, \\ \dot{p} &= s\left(1-h\frac{p}{n}\right) p. \end{cases}$$

$$\begin{cases} \dot{\mathfrak{n}} &= \left(1-\frac{\mathfrak{n}}{\mathfrak{k}} - \mathfrak{h}\frac{\mathfrak{p}}{\mathfrak{n}+1}\right) \mathfrak{n}, \\ \dot{\mathfrak{p}} &= \mathfrak{s}\left(1-\frac{\mathfrak{p}}{\mathfrak{n}}\right) \mathfrak{p}. \end{cases}$$

r, s, e, h, k_1, k_2 parameters.

$\mathfrak{s}, \mathfrak{h}, \mathfrak{k}$ parameters

$$\mathfrak{t} = r\,t,\ \mathfrak{n} = \frac{n}{e},\ \mathfrak{p} = \frac{h\,p}{e},\ \mathfrak{s} = \frac{s}{r},\ \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$$

Parameter reduction [Hubert Labahn FoCM 13]

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r, s, e, h, k_1, k_2 parameters.

Scaling symmetry:

$$\begin{aligned} s &= \eta^{-1}\,\tilde{s}, & r &= \eta^{-1}\,\tilde{r}, & t &= \eta\,\tilde{t}, \\ k_2 &= \eta^{-1}\mu\,\nu^{-1}\,\tilde{k}_2, & d &= \mu\,\tilde{d}, & n &= \mu\,\tilde{n}, \\ k_1 &= \mu\,\tilde{k}_1, & h &= \mu\nu^{-1}\,\tilde{h}, & p &= \nu\,p, \end{aligned}$$

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r, s, e, h, k_1, k_2 parameters.

Scaling symmetry:

$$A = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Parameter reduction

Prey-predator model [Murray, Mathematical Biology (2002)]

$$\begin{cases} \dot{n} & = & \left(\left(1-\frac{n}{k_1}\right) r - k_2 \frac{p}{n+e}\right) n, \\ \dot{p} & = & s\left(1-h\frac{p}{n}\right) p. \end{cases}$$

r, s, e, h, k_1, k_2 parameters.

Hermite multiplier for the matrix defining the Scaling symmetry:

$$A \begin{bmatrix} V_{\mathfrak{i}} & V_{\mathfrak{n}} \end{bmatrix} = \begin{bmatrix} I_r & 0 \end{bmatrix}$$

Invariants :

$$\mathfrak{t} = r\,t,\ \mathfrak{n} = \frac{n}{e},\ \mathfrak{p} = \frac{h\,p}{e},\ \mathfrak{s} = \frac{s}{r},\ \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$$

Rewrite rules : $r \longrightarrow 1, \quad h \longrightarrow 1, \quad k_1 \longrightarrow 1;$

$$s \longrightarrow \mathfrak{s}, \quad k_2 \longrightarrow \mathfrak{k}, \quad d \longrightarrow \mathfrak{d}; \qquad t \longrightarrow \mathfrak{t}, \quad n \longrightarrow \mathfrak{n}, \quad p \longrightarrow \mathfrak{p}.$$

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$$\begin{cases} \dot{\mathfrak{n}} &= \left(1-\frac{\mathfrak{n}}{\mathfrak{k}} - \mathfrak{h}\frac{\mathfrak{p}}{\mathfrak{n}+1}\right) \mathfrak{n}, \\ \dot{\mathfrak{p}} &= \mathfrak{s}\left(1-\frac{\mathfrak{p}}{\mathfrak{n}}\right) \mathfrak{p}. \end{cases}$$

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Avoiding fractional powers

Model for a chemical reaction

$$\begin{cases} \frac{dx}{dt} &= a - k\,x + h\,x^2 y \\ \frac{dy}{dt} &= b - h\,x^2 y \end{cases}$$

[Murray 2002]:

$$\mathfrak{a} = \frac{h^{1/2}}{k^{3/2}}\,a, \quad \mathfrak{b} = \frac{h^{1/2}}{k^{3/2}}\,b;$$
$$\mathfrak{t} = k\,t, \quad \mathfrak{x} = \frac{h^{1/2}}{k^{1/2}}\,x, \quad \mathfrak{y} = \frac{h^{1/2}}{k^{1/2}}\,y$$

$$\begin{cases} \frac{d\mathfrak{x}}{d\mathfrak{t}} &= \mathfrak{a} - \mathfrak{x} + \mathfrak{x}^2\mathfrak{y} \\ \frac{d\mathfrak{y}}{d\mathfrak{t}} &= \mathfrak{b} - \mathfrak{x}^2\mathfrak{y} \end{cases}$$

Avoiding fractional powers

Model for a chemical reaction

$$\begin{cases} \frac{dx}{dt} &= a - k\,x + h\,x^2 y \\ \frac{dy}{dt} &= b - h\,x^2 y \end{cases}$$

[Murray 2002]:

$$\mathfrak{a} = \frac{h^{1/2}}{k^{3/2}}\,a, \quad \mathfrak{b} = \frac{h^{1/2}}{k^{3/2}}\,b;$$
$$\mathfrak{t} = k\,t, \quad \mathfrak{x} = \frac{h^{1/2}}{k^{1/2}}\,x, \quad \mathfrak{y} = \frac{h^{1/2}}{k^{1/2}}\,y$$

$$\begin{cases} \frac{d\mathfrak{x}}{d\mathfrak{t}} &= \mathfrak{a} - \mathfrak{x} + \mathfrak{x}^2\mathfrak{y} \\ \frac{d\mathfrak{y}}{d\mathfrak{t}} &= \mathfrak{b} - \mathfrak{x}^2\mathfrak{y} \end{cases}$$

[HL13]:

$$\mathfrak{b} = \frac{b}{a},\ \mathfrak{h} = \frac{a^2 h}{k^3};$$
$$\mathfrak{t} = k\,t,\ \mathfrak{x} = \frac{k}{a}\,x,\ \mathfrak{y} = \frac{k}{a}\,y.$$

$$\begin{cases} \frac{d\mathfrak{x}}{d\mathfrak{t}} &= 1 - \mathfrak{x} + \mathfrak{h}\,\mathfrak{x}^2\mathfrak{y} \\ \frac{d\mathfrak{y}}{d\mathfrak{t}} &= \mathfrak{b} - \mathfrak{h}\,\mathfrak{x}^2\mathfrak{y} \end{cases}$$

Sections and the computation rational invariants for applications

The slice method

G an algebraic group acting on Ω.

A subspace $\Lambda \subset \Omega$ is a B-**slice** if

- generic orbits intersect Λ
- $B = \{g \in G \mid g \star \Lambda \subset \Lambda\}$
- $g \star \lambda \in \Lambda_{\mathbb{C}} \quad \Rightarrow \quad g \in B_{\mathbb{C}}$

$$f \in \mathbb{R}(\Omega)^G \quad \Rightarrow \quad f_{|\Lambda} \in \mathbb{R}(\Lambda)^B$$

The slice lemma [Sheshadri 62]

The restriction of rational functions on Ω to Λ is an isomorphism of fields:

$$\mathbb{R}(\Omega)^G \xrightarrow{\cong} \mathbb{R}(\Lambda)^B.$$

Illustration on ternary quadrics

Action:

$$\begin{array}{ccc} \mathrm{O}_3(\mathbb{R}) \times \mathrm{S}_3(\mathbb{R}) & \rightarrow & \mathrm{S}_3(\mathbb{R}) \\ (Q, A) & \mapsto & Q^t\, A\, Q \end{array}$$

Slice:

- Section - Diagonal matrices

$$\Lambda = \left\{ \begin{pmatrix} \lambda_1 & . & . \\ . & \lambda_2 & . \\ . & . & \lambda_3 \end{pmatrix} \right\}$$

  For any symmetric matrix A there exists $Q \in \mathrm{O}_3$ s.t $Q\, A\, Q^T \in \Lambda$.

- Subgroup $\mathrm{B}_3 = \mathfrak{S}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$

  $Q^t\, \Lambda\, Q \subset \Lambda$ if

  - $Q = \begin{pmatrix} \pm 1 & . & . \\ . & \pm 1 & . \\ . & . & \pm 1 \end{pmatrix}$
  - Q is a permutation matrix

Illustration on ternary quadrics

Ω = symmetric matrices. Λ = diagonal matrices.

O_3 the orthogonal group $\mathrm{B}_3 = \mathfrak{S}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$

$(\sigma, \epsilon) \in \mathrm{B}_3$ $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda$

$$(\sigma, \epsilon) \star (\lambda_1, \lambda_2, \lambda_3) = \left(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}\right)$$

Invariants of B_3: the Newton sums (or symmetric functions)

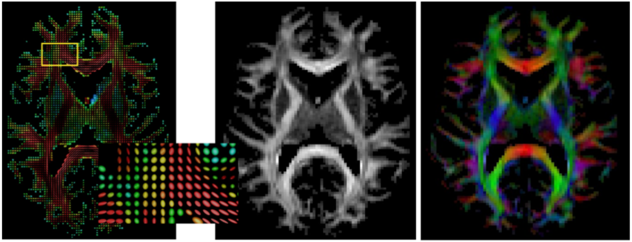
$$p_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad p_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad p_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$$

They are the restrictions of

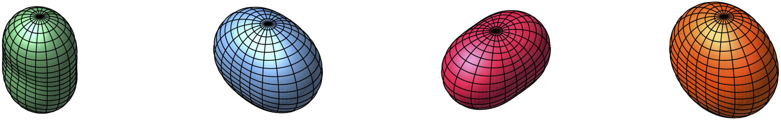
$$q_1 = \mathrm{Tr}(A), \quad q_2 = \mathrm{Tr}(A^2), \quad q_3 = \mathrm{Tr}(A^3)$$

$$\mathbb{R}(\Lambda)^{\mathrm{B}_3} = \mathbb{R}(p_1,\ p_2,\ p_3) \quad \Rightarrow \quad \mathbb{R}(\Omega)^{\mathrm{O}_3} = \mathbb{R}(q_1,\ q_2,\ q_3)$$

Diffusion tensor: a positive symmetric matrix at each voxel



$$f(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \omega_{20} & \frac{1}{2}\omega_{11} & \frac{1}{2}\omega_{10} \\ \frac{1}{2}\omega_{11} & \omega_{02} & \frac{1}{2}\omega_{01} \\ \frac{1}{2}\omega_{10} & \frac{1}{2}\omega_{01} & \omega_{00} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



Biomarkers as invariants under O_3

Mean diffusivity

$$\bar{\lambda} = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}$$

Fractional Anisotropy

$$\sqrt{\frac{(\lambda_1 - \bar{\lambda})^2 + (\lambda_2 - \bar{\lambda})^2 + (\lambda_3 - \bar{\lambda})^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

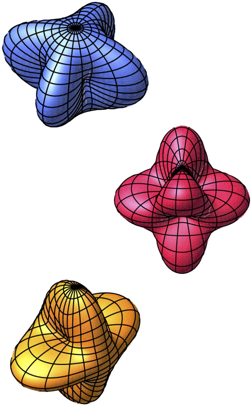
$$\bar{\lambda} = \frac{1}{3}\mathrm{tr}(\omega), \qquad a = \sqrt{1 - \frac{1}{3}\frac{\mathrm{tr}(\omega)^2}{\mathrm{tr}(\omega^2)}} \qquad \text{where } A = \begin{pmatrix} \omega_{20} & \frac{1}{2}\omega_{11} & \frac{1}{2}\omega_{10} \\ \frac{1}{2}\omega_{11} & \omega_{02} & \frac{1}{2}\omega_{01} \\ \frac{1}{2}\omega_{10} & \frac{1}{2}\omega_{01} & \omega_{00} \end{pmatrix}$$

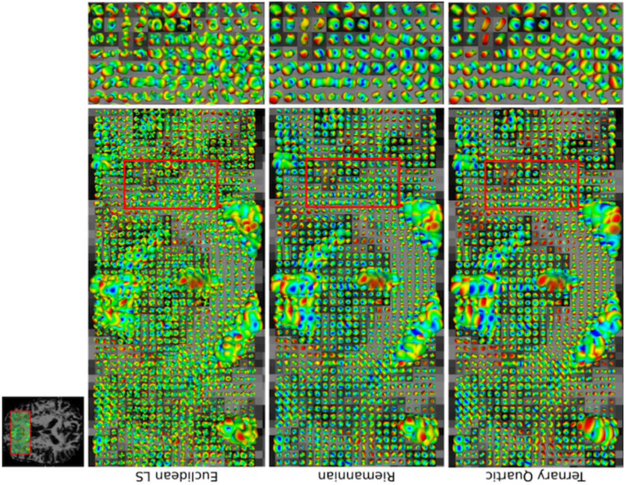
These biomarkers are invariant under the action $(Q, A) \to QAQ^T$,

for $Q \in \mathrm{SO}_3$ a rotation in $\begin{pmatrix} x & y & z \end{pmatrix}$ *space*..

Higher order models: ternary quartics

$$\omega_{40}x^4 + \omega_{31}x^3y + \omega_{22}x^2y^2 + \omega_{13}xy^3 + \omega_{04}y^4 + \ldots\ldots + \omega_{03}yz^3 + \omega_{00}z^4$$





Rotation in 3-space

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a^2+b^2-c^2-d^2 & 2\,bc-2\,ad & 2\,bd+2\,ac \\ 2\,bc+2\,ad & a^2-b^2+c^2-d^2 & 2\,cd-2\,ab \\ 2\,bd-2\,ac & 2\,cd+2\,ab & a^2-b^2-c^2+d^2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}$$

$$a^2+b^2+c^2+d^2=1$$

$$p = \omega_{40}x^4 + \omega_{31}x^3y + \omega_{22}x^2y^2 + \omega_{13}xy^3 + \omega_{04}y^4 + \omega_{30}x^3z + \ldots\ldots + \omega_{00}z^4$$

$$\tilde{p}(x,y,z) = p(\tilde{x},\tilde{y},\tilde{z})$$

$$\tilde{p} = \tilde{\omega}_{40}x^4 + \tilde{\omega}_{31}x^3y + \tilde{\omega}_{22}x^2y^2 + \tilde{\omega}_{13}xy^3 + \tilde{\omega}_{04}y^4 + \tilde{\omega}_{30}x^3z + \ldots\ldots + \tilde{\omega}_{00}z^4$$

Induced action

$$\begin{bmatrix} \tilde{\omega}_{40} \\ \vdots \\ \tilde{\omega}_{00} \end{bmatrix} = R(a,b,c,d) \begin{bmatrix} \omega_{40} \\ \vdots \\ \omega_{00} \end{bmatrix}, \qquad \omega \in \mathbb{R}^{15}, \quad R(a,b,c,d) \text{ of degree } 8$$

The ring of polynomial invariants

$$3\,\omega_{40} + 3\,\omega_{04} + 3\,\omega_{00} + \omega_{22} + \omega_{20} + \omega_{02}$$

$$\begin{aligned}
&25\,(3\,\omega_{30} + \omega_{121} + 3\omega_{10})^2 + 25\,(\omega_{21} + 3\,\omega_{03} + 3\,\omega_{0,1})^2 + 25\,(3\,\omega_{31} + 3\,\omega_{13} + \omega_{11})^2\\
&-4\,(27\omega_{00} - 3\omega_{40} - \omega_{22} - 3\omega_{04} + 4\omega_{20} + 4\omega_{02})\,(27\omega_{04} - \omega_{20} - 3\omega_{40} + 4\omega_{22} + 4\omega_{02} - 3\omega_{00})\\
&-4\,(27\omega_{00} - 3\omega_{40} - \omega_{22} - 3\omega_{04} + 4\omega_{20} + 4\omega_{02})\,(27\omega_{40} + 4\omega_{22} - 3\omega_{04} + 4\omega_{20} - \omega_{02} - 3\omega_{00})\\
&-4\,(27\omega_{04} - 3\omega_{40} + 4\omega_{22} - \omega_{20} + 4\omega_{02} - 3\omega_{00})\,(27\omega_{40} + 4\omega_{22} - 3\omega_{04} + 4\omega_{20} - \omega_{02} - 3\omega_{00})
\end{aligned}$$

At least 12 invariants. . .

A. Ghosh, T. Papadopoulo, and R. Deriche. IEEE International Symposium on Biomedical Imaging, 2012.

A. Ghosh, T. Papadopoulo, and R. Deriche. Computational Diffusion MRI Workshop (CDMRI), MICCAI, 2012.

E. Caruyer, R. Verma. Medical Image Analysis 20:1, 2015.

Auffray, Kolev, Olive. A minimal integrity basis for the elasticity tensor (2017)

Olive. About Gordan's Algorithm for Binary Forms. J. FoCM 2016.

64 polynomial invariants given as transvectants

$$(\,\mathrm{O}_3(\mathbb{R});\ \mathbb{R}[x,y,z]_4)\ \xrightarrow{\cong}\ (\mathrm{SL}_2(\mathbb{C});\ \mathbb{C}[x,y]_8 \oplus \mathbb{C}[x,y]_4)$$

O_3-invariants of quartics for brain imaging

Problem Specifications

- A *complete* set of $k \geq 12$ invariants (answer : $k = 12$)
- How to evaluate them on numerical data.
- What is the image in $\mathbb{R}^{12}$? (what are the possible values)
- Representative in the pre-image of $a \in \mathbb{R}^{12}$? (inverse problem)

Strategy

$$\mathbb{R}\left(\Omega_4\right)^{O_3} \cong \mathbb{R}(\Lambda_4)^{B_3}.$$

Harmonic decomposition of quartics

$$\mathbb{R}[x,y,z]_4 \;=\; \mathcal{H}_4 \oplus (x^2+y^2+z^2)\,\mathbb{R}[x,y,z]_2$$

$$\mathcal{H}_k = \{\ h \in \mathbb{R}[x,y,z]_k \mid \Delta h = 0\ \} \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

B_3-Slice

$$\Omega_4 = \mathbb{R}[x,y,z]_4 \;=\; \mathcal{H}_4 \oplus \mathbb{R}[x,y,z]_2 \qquad \Lambda_4 \;=\; \mathcal{H}_4 \oplus \Lambda_2$$

B_3-equivariant basis for $\mathcal{H}_4$

[Görlach Hubert Papado 19]

$$\begin{aligned}
r_1 &= y^4 - 6\,y^2z^2 + z^4, & t_1 &= 6\,xyz^2 - x^3y - xy^3, & u_1 &= y^3z - yz^3;\\
r_2 &= z^4 - 6\,z^2x^2 + x^4, & t_2 &= 6\,yzx^2 - y^3z - yz^3, & u_2 &= z^3x - zx^3,\\
r_3 &= x^4 - 6\,x^2y^2 + y^4, & t_3 &= 6\,zxy^2 - z^3x - zx^3, & u_3 &= x^3y - xy^3.
\end{aligned}$$

The B_3-equivariance of the basis for $\Lambda_4 = \mathcal{H}_4 \oplus \Lambda_2$

$$v = (\rho_1 r_1 + \rho_2 r_2 + \rho_3 r_3) + (\tau_1 t_1 + \tau_2 t_2 + \tau_3 t_3) + (\mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3) + \mathrm{q}(\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2)$$

$$(\sigma, \epsilon) \in B_3, \qquad \sigma \in \mathfrak{S}_3, \qquad \epsilon = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \qquad \tilde{v} = (\sigma, \epsilon) \star v$$

$$\tilde{v} = (\tilde{\rho}_1 r_1 + \tilde{\rho}_2 r_2 + \tilde{\rho}_3 r_3) + (\tilde{\tau}_1 t_1 + \tilde{\tau}_2 t_2 + \tilde{\tau}_3 t_3) + (\tilde{\mu}_1 u_1 + \tilde{\mu}_2 u_2 + \tilde{\mu}_3 u_3) + \mathrm{q}(\tilde{\lambda}_1 x^2 + \tilde{\lambda}_2 y^2 + \tilde{\lambda}_3 z^2)$$

$$\begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ \tilde{\lambda}_3 \end{bmatrix} = P_\sigma \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_3 \end{bmatrix} = P_\sigma \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \\ \tilde{\tau}_3 \end{bmatrix} = |\epsilon|\, \epsilon\, P_\sigma \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \tilde{\mu}_3 \end{bmatrix} = |\epsilon||P_\sigma| \epsilon P_\sigma \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

B_3 invariants [Görlach Hubert Papadopoulos 19]

$$\begin{bmatrix}\tilde\lambda_1\\ \tilde\lambda_2\\ \tilde\lambda_3\end{bmatrix} = P_\sigma \begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}, \quad \begin{bmatrix}\tilde\rho_1\\ \tilde\rho_2\\ \tilde\rho_3\end{bmatrix} = P_\sigma \begin{bmatrix}\rho_1\\ \rho_2\\ \rho_3\end{bmatrix}, \quad \begin{bmatrix}\tilde\tau_1\\ \tilde\tau_2\\ \tilde\tau_3\end{bmatrix} = |\epsilon|\,\epsilon\, P_\sigma \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}, \quad \begin{bmatrix}\tilde\mu_1\\ \tilde\mu_2\\ \tilde\mu_3\end{bmatrix} = |\epsilon||P_\sigma|\epsilon P_\sigma \begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix}$$

Apply [Hubert & Kogan JSC 07]

Minimal generating set: 12 invariants

$$\tau_1^2+\tau_2^2+\tau_3^2, \quad \tau_1^2\tau_2^2+\tau_2^2\tau_3^2+\tau_3^2\tau_1^2, \quad \tau_1\tau_2\tau_3.$$

and the entries of

$$\begin{bmatrix}1 & 1 & 1\\ \tau_1^2 & \tau_2^2 & \tau_3^2\\ \tau_1^4 & \tau_2^4 & \tau_3^4\end{bmatrix} \begin{bmatrix}\lambda_1 & \rho_1 & |\lambda|\mu_1\tau_1\\ \lambda_2 & \rho_2 & |\lambda|\mu_2\tau_2\\ \lambda_3 & \rho_3 & |\lambda|\mu_3\tau_3\end{bmatrix}$$

O_3-invariants : evaluation

$p_1, \ldots, p_{12}$ form a generating set of B_3-invariants on the slice Λ_4

$q_1, \ldots, q_{12}$ are the O_3-invariants on Ω_4

uniquely determined by their restrictions $p_1, \ldots, p_{12}$

To evaluate $q_1, \ldots, q_{12}$, their expressions are not needed!

In: $(\rho, \tau, \mu, \omega) \in \Omega_4 = \mathcal{H}_4 \oplus \mathbb{R}[x, y, z]_2$

1. Compute $Q \in \mathrm{O}_3$ s.t. $Q \begin{bmatrix} \omega_{11} & \frac{1}{2}\omega_{12} & \frac{1}{2}\omega_{13} \\ \frac{1}{2}\omega_{12} & \omega_{22} & \frac{1}{2}\omega_{23} \\ \frac{1}{2}\omega_{13} & \frac{1}{2}\omega_{23} & \omega_{33} \end{bmatrix} Q^T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$
2. $(\tilde{\rho}, \tilde{\tau}, \tilde{\mu}, \lambda, 0) := Q \star (\rho, \tau, \mu, \omega)$

Out: $q(\rho, \tau, \mu, \omega) := p(\tilde{\rho}, \tilde{\tau}, \tilde{\mu}, \lambda)$

O_3-invariants : Inverse problem

Given $(c_1, \ldots, c_{12}) \in \mathbb{R}^{12}$,
how to find $(\rho, \tau, \mu, \omega) \in \mathcal{H}_4 \oplus \mathbb{R}[x, y, z]_2$
such that $q_i(\rho, \tau, \mu, \omega) = c_i$?

We can look for $(\rho, \tau, \mu, \lambda) \in \mathcal{H}_4 \oplus \Lambda_2$ such that $p_i(\rho, \tau, \mu, \lambda) = c_i$.

1. $\tau_1^2, \tau_2^2, \tau_3^2$ are the roots of $\tau^3 - c_1\tau^2 + c_2\tau - {c_3}^2$

$$p_1 = \tau_1^2 + \tau_2^2 + \tau_3^2, \quad p_2 = \tau_1^2\tau_2^2 + \tau_2^2\tau_3^2 + \tau_3^2\tau_2^2, \quad p_3 = \tau_1\tau_2\tau_3.$$

We can also make explicit the conditions for τ_1, τ_2, τ_3 to be real

2. Solve the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ \tau_1^2 & \tau_2^2 & \tau_3^2 \\ \tau_1^4 & \tau_2^4 & \tau_3^4 \end{bmatrix} \begin{bmatrix} \lambda_1 & \rho_1 & |\lambda|\mu_1\tau_1 \\ \lambda_2 & \rho_2 & |\lambda|\mu_2\tau_2 \\ \lambda_3 & \rho_3 & |\lambda|\mu_3\tau_3 \end{bmatrix} = \begin{bmatrix} c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \\ c_{10} & c_{11} & c_{12} \end{bmatrix}$$

[Hubert, Görlach & Papadopoulo; FoCM 2019]

- A generating set of (functionally) independent 12 *rational* invariants
  - uniquely characterized by their restrictions to a *slice*
  - which are trinomials invariant under B_3.
- A robust numerical algorithm to evaluate them
  - Diagonalize a 3×3 symmetric matrix
- Complete solution to the inverse problem
  - Roots of a degree 3 polynomial
  - Solve 3×3 linear systems $Ax_i = b_i$, $i = 1, 2, 3$
- A rewriting algorithm
- Also for sextics, octics, ... All even degree ternary forms.

Next

- Describe the orbit space of positive quartics
- Practical results on synthetic and actual data.
- Does the strategy apply to the action on integrated integrals

- Rational Invariants. Construction and Rewriting.
  H. & Kogan, J. of Symbolic Coputation (2007)
- Smooth and Algebraic Invariants. Local and Global Constructions
  H. & Kogan, Foundations of Computational Mathematics (2007)
- Linear actions of $(\mathbb{K}^*)^m$ and parameter reduction.
  H. & Labahn, Foundations of Computational Mathematics (2013)
- $O(3)$ on $\mathbb{K}[x, y, z]_{2d}$ and neuroimaging
  Görlach, H. & Papadopoulo, Foundations of Computational Math. (2019)
- Scaling invariants and parameter reduction in PDEs. in preparation

Finite groups:

- Linear actions of finite abelian groups and solving polynomial systems:
  H. & Labahn, Mathematics of Computation (2016)
- Fundamental invariants and equivariants of finite groups
  H. & Rodriguez Bazan [https://hal.inria.fr/hal-03209117]