WEIL SPACES AND THE EMBEDDING THEOREM FOR TANGENT CATEGORIES

1) THE FREE TANGENT CATY ON AN OBJECT

Notation: if $\mathcal{C}$ is a tangent caty, then we call the fibre products $T_n X = TX \times_{p_X} TX \times_{p_X} \cdots \times_{p_X} TX$ and the pullbacks

$$\begin{tikzcd} T_2 X \arrow[r, "m"] \arrow[d, "p \cdot \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2 X \arrow[d, "Tp"] \\ X \arrow[r, "0"'] & TX \end{tikzcd}$$

, <u>tangent limits</u>.

<u>Defn</u> A map of tangent catys $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor that preserves tangent limits and comes equipped with $\varphi : T_{\mathcal{D}} F \cong F T_{\mathcal{C}}$, coherently with the rest of data.

<u>Defn</u> $\underline{\mathrm{Tang}}$ is the 2-caty of tangent catys, tangent functors + "tangent transformations".

So we can speak of "the free tangent caty on an object".

<u>Def^n</u> • Let k be a commutative rig. $\underline{k\text{-alg}}$ is the tangent caty of k-algebras (comm), with tangent functor

$$T(A) = A[x]/_{x^2}$$

and with

$$p_A : T(A) \longrightarrow A$$
$$a + bx \longmapsto a.$$

Alternatively, $T(A) = W \otimes A$ where $W = k[x]/_{x^2}$.

It follows that $T_n(A) = W_n \otimes A$ where $W_n = k[x_1, \dots, x_n] / (x_i x_j)_{i \leq j}$

- $\underline{\mathrm{Weil}}_1$ is the full sub-tangent-caty of k-alg generated by k. Explicitly, it's the full subcaty on $W_{n_1} \otimes \cdots \otimes W_{n_k}$.

The nomenclature comes from SDG: a <u>Weil algebra</u> is a comm. k-alg A st underlying k-module is f.g. free, and st $A = k1 \oplus M$ where M is composed of nilpotents. These comprise a sub-tangent-caty $\underline{\mathrm{Weil}}$ of k-alg, and now $\underline{\mathrm{Weil}}_1 \subseteq \underline{\mathrm{Weil}}$.

<u>THM</u> (Leung) $\underline{\mathrm{Weil}}_1$ is the free tang. caty on one object (k).

<u>Proof</u> (idea) If $\mathcal{C}$ a tang. caty, $X \in \mathcal{C}$, define

$$
\begin{array}{rcl}
\llcorner X \lrcorner : \underline{\mathrm{Weil}}_1 & \longrightarrow & \mathcal{C} \\
k & \longmapsto & X \\
W = T(k) & \longmapsto & TX \\
W \otimes W = T^2(k) & \longmapsto & TTX \\
W_2 = T_2(k) & \longmapsto & T_2 X \\
W_{n_1} \otimes \cdots \otimes W_{n_k} & \longmapsto & T_{n_k} \cdots T_{n_1} X \\
p_k : W \longrightarrow k & \longmapsto & p_X : TX \longrightarrow X \\
0 & & \\
\ell & & \\
\vdots & &
\end{array}
$$

\+ lot of hard work.

$\square$

2) <u>WEIL SPACES</u>

Can see Weil algebras as certain affine schemes, namely, the function algebras on infinitesimal neighborhoods of a point. It is natural to want to glue such things together.

Defn

(1) The caty WeilSp of Weil spaces is the full subcaty of [$\underline{\text{Weil}}$, $\underline{\text{Set}}$] on pb-preserving functors

(2) $\cdot$ $\cdot$ $\cdot$ $\text{Weil}_1 Sp$ $\cdot$ Weil_1 $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\underline{\text{Weil}}_1$ $\cdot$ $\cdot$ $\cdot$ tangent limit $\cdot$ $\cdot$ $\cdot$ $\cdot$

The Yoneda embeddings $\text{Weil}_{(1)}^{op} \xrightarrow{Y} \text{Weil}_{(1)} Sp$, exhibit $\text{Weil}_{(1)} Sp$ as free cocomp of $\text{Weil}_{(1)}$ preserving existing pushouts / tangent colimits.

LEMMA The cart. product in $\text{Weil}_{(1)} Sp$ has the property that

$$\frac{X \times Y \longrightarrow Z \quad \text{in } \text{Weil}_{(1)} Sp}{X(A) \times Y(B) \longrightarrow Z(A \otimes B) \quad \text{nat in } A, B} \quad (*)$$

Proof Since $\otimes$ is coproduct in $\underline{\text{Weil}}_{(1)}$, have $yA \times yB \cong y(A \otimes B)$. Now write X, Y as colimits of reps., use that $\times$ in WeilSp pres colimits in each variable; $(*)$; and Yoneda lemma. □

3) TANGENT CATYS AS ENRICHED CATYS

There is a standard notion of caty $\mathcal{C}$ enriched over a sym. monoidal caty $\mathcal{V}$: involves a set of objects; hom-objects $\mathcal{C}(X,Y) \in \mathcal{V}$; composition $\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$; identities $k \longrightarrow \mathcal{C}(X,X)$; plus axioms.

Defn If $\mathcal{C}$ is a $\mathcal{V}$-enr. caty, $X \in \mathcal{C}$, $V \in \mathcal{V}$, then the power of X by V is an object $V \pitchfork X \in \mathscr{C}$ t/w $\mathscr{V}$-natural bijection

$$\frac{W \longrightarrow \mathscr{C}(Y, V \pitchfork X) \quad \text{in } \mathscr{V}}{V \otimes W \longrightarrow \mathscr{C}(Y, X) \quad \text{in } \mathscr{V}}$$

THM$^{(G.)}$ A tangent caty $\mathscr{C}$ is the same as a (Weil$_1$, Sp, $\times$)-enriched caty with powers by representables.

Proof Explicit Leung's result: given a tang. caty $\mathscr{C}$, define $\underline{\text{Weil}_1}$, Sp enriched caty $\underline{\mathscr{C}}$ with same objects, and $\underline{\mathscr{C}}(X, Y)$ given by

$$\underline{\text{Weil}_1} \xrightarrow{\lfloor Y \rfloor} \mathscr{C} \xrightarrow{\mathscr{C}(X, -)} \text{Set}$$

ie $\underline{\mathscr{C}}(X, Y)(W_{n_1} \otimes \cdots \otimes W_{n_k}) = \mathscr{C}(X, T_{n_k} \cdots T_{n_1} Y)$.

For composition; need $\underline{\mathscr{C}}(Y, Z) \otimes \underline{\mathscr{C}}(X, Y) \longrightarrow \underline{\mathscr{C}}(X, Z)$.

By lemma, same as giving

$$\mathscr{C}(Y, Z)(A) \times \mathscr{C}(X, Y)(B) \longrightarrow \mathscr{C}(X, Z)(A \otimes B)$$

eg. $\quad \mathscr{C}(Y, Z)(W) \times \mathscr{C}(X, Y)(W_2) \longrightarrow \mathscr{C}(X, Z)(W \otimes W_2)$

$$(Y \xrightarrow{g} TZ \;,\; X \xrightarrow{f} T_2 Y) \longmapsto X \xrightarrow[f]{} T_2 Y \xrightarrow[T_2 g]{} T_2 T Z.$$

Powers by reps: $yW \pitchfork X$ characterised by:

$$\frac{y(k) \longrightarrow \underline{\mathscr{C}}(Y, yW \pitchfork X)}{y(W) = y(k) \otimes y(W) \longrightarrow \underline{\mathscr{C}}(Y, X)} \qquad \text{so take } yW \pitchfork X = TX.$$

... $\square$

4) THE EMBEDDING THM FOR TANGENT CATYS

A tangent caty $\mathcal{C}$ is representable if it's cartesian closed and $T_n X = X^{D_n}$ for suitable objects $D_n \in \mathcal{C}$.

THM (G.) Every small tangent caty $\mathcal{C}$ embeds fully into a representable tangent caty.

Proof View $\mathcal{C}$ as a Weil$_1$Sp-caty; consider enriched Yoneda embedding

$$\mathcal{C} \longrightarrow [\mathcal{C}^{op}, \text{Weil}_1\text{Sp}].$$

This corresponds to a full tangent embedding. Moreover, the codomain is a cart. closed caty (since presheaves on a c.c. base). Moreover, tang. structure on codomain is

$$yW \pitchfork (-) = (-)^{yW \cdot 1}$$ copower of $1 \in \text{Weil}_1\text{Sp}$ by yW

So representable. $\square$

We can unravel above construction to find our embedding is of $\mathcal{C}$ into a caty of tangent profunctors $\mathcal{C} \nrightarrow \text{Weil}_1$