Frobenius-Eilenberg-Moore objects in dagger 2-categories

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A note on Frobenius-Eilenberg-Moore objects in dagger 2-categories
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References

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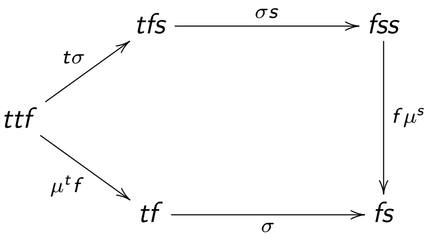
- “The formal theory of monads I & II” - R. Street and S. Lack (1972, 2002)
- “Frobenius algebras and ambidextrous adjunctions” - A. Lauda (2006)
- “Frobenius monads and pseudomonoids” - R. Street (2004)

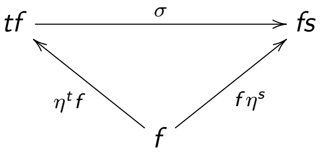
The formal definition of monads

The *formal definition of monads* due to Benábou (1967).

A *monad* in a 2-category $\mathcal{K}$ is a monoid object $(A, s, \mu, \eta) = (A, s)$ in the category $\mathcal{K}(A, A)$, for some $A \in \mathcal{K}$.

A *morphism of monads* $(f, \sigma) : (A, s) \longrightarrow (D, t)$ consists of a 1-cell $f : A \longrightarrow D$ and a 2-cell $\sigma : tf \longrightarrow fs$ in $\mathcal{K}$ making the diagrams below commute





The formal definition of monads, cont.

A *monad morphism transformation* $(A, s) \underset{(g,\gamma)}{\overset{(f,\sigma)}{\Downarrow \alpha}} (D, t)$ is a 2-cell $\alpha : f \longrightarrow g$ in $\mathcal{K}$, such that the diagram below commutes

$$\begin{array}{ccc} tf & \xrightarrow{\;t\alpha\;} & tg \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \gamma} \\ fs & \xrightarrow[\;\alpha s\;]{} & gs \end{array}$$

Equivalently: A monad in a 2-category $\mathcal{K}$ is a lax functor $\mathbf{1} \longrightarrow \mathcal{K}$ from the terminal 2-category $\mathbf{1}$ to $\mathcal{K}$.

For each 2-category $\mathcal{K}$, this defines a 2-category

$$\mathrm{Mnd}(\mathcal{K}) = \mathrm{LaxFun}(\mathbf{1}, \mathcal{K})$$

EM-objects

Eilenberg-Moore objects (Street, 1972)

For each monad (A, s) in a 2-category $\mathcal{K}$, there is a 2-functor $\mathcal{K}^{\mathrm{op}} \longrightarrow \mathrm{Cat} : X \mapsto \mathcal{K}(X, A)^{\mathcal{K}(X,s)}$. If this 2-functor is representable, A^s is denoted as the representing object, and is called the *Eilenberg–Moore (EM) object* of the monad (A, s).

That is,

$$\mathcal{K}(X, A^s) \cong \mathcal{K}(X, A)^{\mathcal{K}(X,s)}$$

2-naturally in the arguments.

Example: in 2-category Cat of categories, functors and natural transformations, EM-objects are usual Eilenberg-Moore categories for the monad.

Free completion under EM-objects

EM objects are weighted limits (Street, 1976) $\Longrightarrow$ free completion under EM objects.

Theorem (Street)

For a 2-category $\mathcal{K}$, there is a 2-category $\mathsf{EM}(\mathcal{K})$ having Eilenberg-Moore objects and a fully faithful 2-functor $Z : \mathcal{K} \longrightarrow \mathsf{EM}(\mathcal{K})$ with the property that for any 2-category $\mathcal{L}$ with Eilenberg–Moore objects, composition with Z induces an equivalence of categories:

$$[\mathsf{EM}(\mathcal{K}), \mathcal{L}]_{\mathsf{EM}} \approx [\mathcal{K}, \mathcal{L}]$$

Free completion, cont.

The Eilenberg-Moore completion can also be given an explicit description (Street-Lack, 2002). $\mathrm{EM}(\mathcal{K})$ has:

- objects as monads (A, s) of $\mathcal{K}$
- 1-cells as morphisms of monads $(u, \phi) : (A, s) \longrightarrow (B, t)$
- 2-cells $\rho : (u, \phi) \longrightarrow (v, \psi)$ as 2-cells ρ in $\mathcal{K}$ suitably commuting with a specified "Kleisli composition".

In general, $\mathrm{EM}(\mathcal{K}) \not\approx \mathrm{Mnd}(\mathcal{K})$

But: $E : \mathrm{Mnd}(\mathcal{K}) \longrightarrow \mathrm{EM}(\mathcal{K})$, which is identity on 0- and 1-cells

Examples of EM($\mathcal{K}$)

Example: EM(Cat)

- objects as usual monads $(\boldsymbol{X}, T, \mu, \eta)$
- 1-cells as pairs $(F, \overline{F})$ of functors making the diagram below commute

$$\begin{array}{ccc} \boldsymbol{X}^T & \xrightarrow{\overline{F}} & \boldsymbol{Y}^S \\ {\scriptstyle U^T}\downarrow & & \downarrow{\scriptstyle U^S} \\ \boldsymbol{X} & \xrightarrow[F]{} & \boldsymbol{Y} \end{array}$$

- 2-cells $\sigma : (F, \overline{F}) \longrightarrow (G, \overline{G})$ as natural transformation $\sigma : \overline{F} \longrightarrow \overline{G}$

Examples of EM($\mathcal{K}$)

Example: One-object 2-category $\Sigma(\mathbf{Vect})$ = the suspension and strictification of **Vect**. In EM($\Sigma(\mathbf{Vect})$):

- objects are the usual algebras A from linear algebra
- 1-cells are pairs $(V,\phi) : A_1 \longrightarrow A_2$, with V a vector space and $\phi : V \otimes A_2 \longrightarrow A_1 \otimes V$ a linear map which form a *left-free bimodule*
- 2-cells $(V,\phi) \longrightarrow (V',\phi') : A_1 \longrightarrow A_2$ are linear maps $\rho : V \longrightarrow A_1 \otimes V'$ which are bimodule homomorphisms of left-free bimodules

Frobenius monads

A monad (X, t, μ, η) in a 2-category $\mathcal{K}$ is called a *Frobenius monad* if there exists a comonad (X, t, δ, ϵ) such that the *Frobenius law* is satisfied:

$$t\mu \cdot \delta t = \delta \cdot \mu = \mu t \cdot t\delta$$

Example: A Frobenius monad in $\Sigma(\mathbf{Vect}_k)$ is just the usual notion of a Frobenius algebra; that is, an k-algebra A equipped with a nondegenerate bilinear form $\sigma : A \times A \longrightarrow k$ that satisfies:

$$\sigma(ab, c) = \sigma(a, bc)$$

Frobenius monads, cont.

Theorem (Lauda, 2006)

For 1-cells $f : A \longrightarrow B$ and $u : B \longrightarrow A$ in a 2-category $\mathcal{K}$, if $f \dashv u \dashv f$ is an ambidextrous adjunction, then the monad uf generated by the adjunction is a Frobenius monad.

Corollary (Lauda, 2006)

Given a Frobenius monad (X, t, μ, η) a 2-category $\mathcal{K}$, in $\mathrm{EM}(\mathcal{K})$ the left adjoint $f^t : X \longrightarrow X^t$ to the forgetful 1-cell $u^t : X^t \longrightarrow X$ is also right adjoint to u^t. Hence, the Frobenius monad (X, t, μ, η) is generated by an ambidextrous adjunction in $\mathrm{EM}(\mathcal{K})$.

Characterising Frobenius algebras

Corollary

For a monoidal category M, each Frobenius object in M arises from an ambidextrous adjunction in $\mathrm{EM}(\Sigma(M))$.

Corollary

*Every Frobenius algebra in the category **Vect** arises from an ambidextrous adjunction in the 2-category whose objects are algebras, morphisms are bimodules of algebras, and whose 2-morphisms are bimodule homomorphisms.*

Corollary

Every 2D topological quantum field theory, in the sense of Atiyah, arises from an ambidextrous adjunction in the 2-category whose objects are algebras, morphisms are bimodules of algebras, and whose 2-morphisms are bimodule homomorphisms.

Characterising Frobenius algebras, cont.

Question: Under appropriate conditions, can we more directly characterize Frobenius objects in a monoidal category? That is, via construction?

- Given a Frobenius monad, can we define an appropriate notion of a "Frobenius-Eilenberg-Moore object"?
- Can we describe FEM-objects as some kind of limit as well as the completion of a 2-category under such FEM-objects like is done for the EM construction?
- Is there an explicit description of this FEM-completion similar to the EM-completion?

Frobenius categories

Theory of accessible categories: A category $\boldsymbol{C}$ is *accessible* if it is equivalent to $\mathrm{Ind}(\boldsymbol{S})$ for some category $\boldsymbol{S}$.

Theory of locally connected categories: A category $\boldsymbol{C}$ is *locally connected* if it is equivalent to $\mathrm{Fam}(\boldsymbol{S})$ for some category $\boldsymbol{S}$.

Question: Can we develop the theory of *Frobenius categories*, i.e. A category $\boldsymbol{C}$ is *Frobenius* if it is equivalent to $\mathrm{FEM}(\boldsymbol{S})$ for some category $\boldsymbol{S}$.

Wreaths

A *wreath* $((A, t), (s, \lambda), \sigma, \nu)$ is an object of $\mathsf{EM}(\mathsf{EM}(\mathcal{K}))$.

Examples: The crossed product of Hopf algebras, factorization systems on categories.

EM is an endo-2-functor 2-Cat $\longrightarrow$ 2-Cat, the universal property of the EM construction determines a 2-functor

$$\mathsf{wr}_{\mathcal{K}} : \mathsf{EM}(\mathsf{EM}(\mathcal{K})) \longrightarrow \mathsf{EM}(\mathcal{K})$$

called the *wreath product*, and there is the embedding 2-functor

$$\mathsf{id}_{\mathcal{K}} : \mathcal{K} \longrightarrow \mathsf{EM}(\mathcal{K})$$

sending objects in $\mathcal{K}$ to the identity monad on them. In total $(\mathsf{EM}, \mathsf{wr}, \mathsf{id})$ is a 2-monad.

Frobenius wreaths

A wreath $((A,t),(s,\lambda),\sigma,\nu)$ in a 2-category $\mathcal{K}$ is called *Frobenius* when, considered as a monad in $\mathsf{EM}(\mathcal{K})$, it is a Frobenius monad.

Theorem (Street, 2004)

The wreath product of a Frobenius wreath on a Frobenius monad is Frobenius.

For our proposed FEM construction and its universal property, this result is immediate since:

$$\mathrm{wr}_{\mathcal{D}} : \mathsf{FEM}(\mathsf{FEM}(\mathcal{D})) \longrightarrow \mathsf{FEM}(\mathcal{D})$$

Dagger categories

A *dagger category* $\boldsymbol{D}$ is a category with an involutive functor $\dagger : \boldsymbol{D}^{\mathrm{op}} \longrightarrow \boldsymbol{D}$ which is the identity on objects. A *dagger functor* between dagger categories is a functor which preserves the daggers.

A *monoidal dagger category* is a dagger category that is also a monoidal category, satisfying $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ and, whose coherence morphisms are *unitary*.

Examples:

- Any groupoid, with $f^\dagger = f^{-1}$.
- The category **Hilb** of complex Hilbert spaces and bounded linear maps, taking the dagger of $f : A \longrightarrow B$ to be its adjoint, i.e. the unique linear map $f^\dagger : B \longrightarrow A$ satisfying $\langle f(a), b\rangle = \langle a, f^\dagger(b)\rangle$ for all $a \in A$ and $b \in B$.

Dagger 2-categories

A 2-category $\mathcal{D}$ is a *dagger 2-category* when each of the hom-categories $\mathcal{D}(A, B)$ are not only categories, but dagger categories, and whose horizontal and vertical composition operators commute with daggers.

Example: The dagger 2-category DagCat of dagger categories, dagger functors and natural transformations.

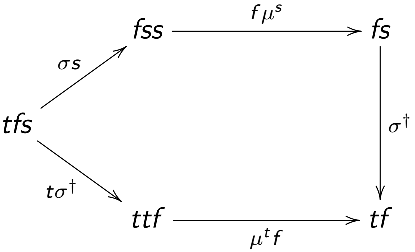
A 2-functor is a *dagger 2-functor* when each of its component functors are dagger functors.

Dagger Frobenius monads

A monad (D, t, μ, η) in a dagger 2-category $\mathcal{D}$ is called a *dagger Frobenius monad* (Heunen and Karvonen, 2016) if the Frobenius law is satisfied:

$$t\mu \cdot \mu^\dagger t = \mu^\dagger \cdot \mu = \mu t \cdot t\mu^\dagger$$

A *morphism of dagger Frobenius monads* $(f, \sigma) : (A, s) \longrightarrow (D, t)$ is a morphism of the underlying monads such that the following diagram commutes:



Dagger Frobenius monads, cont

A *dagger Frobenius monad morphism transformation*

$$(A, s) \underset{(g,\gamma)}{\overset{(f,\sigma)}{\Downarrow \alpha}} (D, t)$$

is a monad morphism transformation of the underlying morphisms of monads, such that the diagram below commutes

$$\begin{array}{ccc} tg & \xrightarrow{t\alpha^\dagger} & tf \\ \gamma \downarrow & & \downarrow \sigma \\ gs & \xrightarrow[\alpha^\dagger s]{} & fs \end{array}$$

For each dagger 2-category $\mathcal{D}$, this defines a dagger 2-category $\mathrm{DFMnd}(\mathcal{D})$.

Examples

For a monoidal dagger category $\boldsymbol{D}$, a dagger Frobenius monad in the dagger 2-category $\Sigma(\boldsymbol{D})$ is called a *dagger Frobenius monoid* in $\boldsymbol{D}$.

Example: Let $\boldsymbol{G}$ be a finite groupoid, and G its set of objects. The assignments

$$1 \longmapsto \sum_{A \in G} \mathrm{id}_A \qquad f \otimes g \longmapsto \begin{cases} f \cdot g & \text{if } f \cdot g \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

define a dagger Frobenius monoid in **FHilb**. Any dagger Frobenius monoid in **FHilb** is of this form.

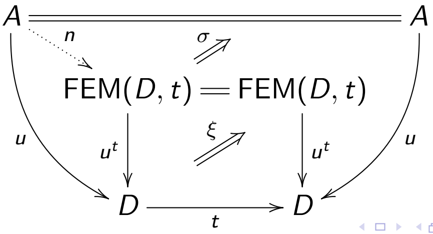
Dagger lax functors

A *dagger lax functor* $F : \mathcal{D} \longrightarrow \mathcal{C}$ between dagger 2-categories is a lax functor satisfying an additional *Frobenius axiom*...

Equivalently: A dagger Frobenius monad in a dagger 2-category $\mathcal{D}$ is a dagger lax functor $\mathbf{1} \longrightarrow \mathcal{D}$ from the terminal 2-category $\mathbf{1}$ to $\mathcal{D}$. So

$$\mathrm{DFMnd}(\mathcal{D}) = \mathrm{DagLaxFun}(\mathbf{1}, \mathcal{D})$$

Dagger lax-natural transformations, dagger lax modifications, dagger lax limits,...



FEM algebras

A *Frobenius-Eilenberg-Moore algebra* for a dagger Frobenius monad (T, μ, η) on a dagger category $\mathbf{D}$ is an Eilenberg-Moore algebra (D, δ) for T, such that the diagram

$$\begin{array}{ccc} T(D) & \xrightarrow{T(\delta^\dagger)} & T^2(D) \\ {\scriptstyle \mu_D^\dagger}\downarrow & & \downarrow{\scriptstyle \mu_D} \\ T^2(D) & \xrightarrow[T(\delta)]{} & T(D) \end{array}$$

commutes. Frobenius-Eilenberg-Moore algebras and homomorphisms of Eilenberg-Moore algebras between FEM-algebras form a dagger category, denoted FEM($\mathbf{D}$, T).

FEM-objects

Frobenius-Eilenberg-Moore objects

For each dagger Frobenius monad (D, t) in a dagger 2-category $\mathcal{D}$, there is a dagger 2-functor

$$\begin{aligned} \mathcal{D}^{\mathsf{op}} &\longrightarrow \mathsf{DagCat} \\ X &\longmapsto \mathsf{FEM}(\mathcal{D}(X, D), \mathcal{D}(X, t)) \end{aligned}$$

If this dagger 2-functor is representable, $\mathsf{FEM}(D, t)$ is denoted as the representing object, and is called the *Frobenius-Eilenberg–Moore (FEM) object* of (D, t).

That is,

$$\mathcal{D}(X, \mathsf{FEM}(D, t)) \cong \mathsf{FEM}(\mathcal{D}(X, D), \mathcal{D}(X, t))$$

dagger 2-naturally in the arguments.

Example

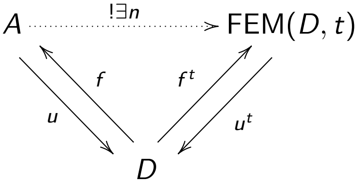
Theorem

Suppose (T, μ, η) is a dagger Frobenius monad on the dagger category $\boldsymbol{D}$. Then $\mathrm{FEM}(\boldsymbol{D}, T)$ *is Frobenius-Eilenberg-Moore object for T.*

Universal property

Theorem

Suppose (D, t) generated by the adjunction $f \dashv u : D \longrightarrow A$ has a FEM-object. Then, there exists a unique 1-cell $n : A \longrightarrow \mathrm{FEM}(D, t)$ – called the right comparison 1-cell – such that $u^t n = u$ and $nf = f^t$.



FK objects

Frobenius-Kleisli objects

A Frobenius-Kleisli object for a dagger Frobenius monad (D, t) in a dagger 2-category $\mathcal{D}$ is a Frobenius-Eilenberg-Moore object for (D, t) in $\mathcal{D}^{\mathsf{op}}$. A Frobenius-Kleisli object for (D, t) is denoted by $\mathsf{FK}(D, t)$, and satisfies the following isomorphism of dagger categories

$$\mathcal{D}(\mathsf{FK}(D, t), X) \cong \mathsf{FEM}(\mathcal{D}(D, X), \mathcal{D}(t, X))$$

2-natural in each of the arguments.

FK objects, cont.

Theorem

Each dagger Frobenius monad $T = (T, \mu, \eta)$ on a dagger category $\boldsymbol{D}$ has a Frobenius-Kleisli object, which is the Kleisli category $\boldsymbol{D}_T$ of the monad T.

Free cocompletions

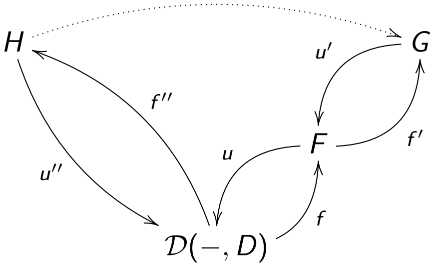
Kelly (2005) provides very general theory of cocompletions, or *closure*, under certain classes of colimits. Hard (impossible?) to reasonably transfer to the dagger context (e.g. Karvonen, 2019)

Build closure $\overline{\mathcal{K}}$ via transfinite process: take $[\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ and start with representables. At each stage, add colimits of the previous stage.

Plan: imitate this for Frobenius-Kleisli objects without general theory, prove universal property similar to that of EM construction.

Free cocompletions, cont.

Transfinite process ends in after one step. **Proof**: In $[\mathcal{D}^{\mathrm{op}}, \mathrm{DagCat}]$



$\mathrm{FK}(\mathcal{D})$ is replete, full dagger-sub-2-category of $[\mathcal{D}^{\mathrm{op}}, \mathrm{DagCat}]$ of objects resulting from the single step. Each representable $\mathcal{D}(-, D)$ is an FK-object for a dagger Frobenius monad on a representable and every object of this dagger 2-category is an FK-object for a dagger Frobenius monad on a representable.

Explicit definition

We want $\mathsf{FEM}(\mathcal{D}) = \mathsf{KL}(\mathcal{D}^{op})^{op}$. So we define $\mathsf{FEM}(\mathcal{D})$ as:

- 0-cells are dagger Frobenius monads in $\mathcal{D}$
- 1-cells are the same as 1-cells in $\mathsf{DFMnd}(\mathcal{D})$
- A 2-cell $(f, \sigma) \longrightarrow (g, \gamma) : (D, t) \longrightarrow (C, s)$ is a 2-cell $\alpha : f \longrightarrow gt$ in $\mathcal{D}$ suitably commuting with a specified "Kleisli composition".

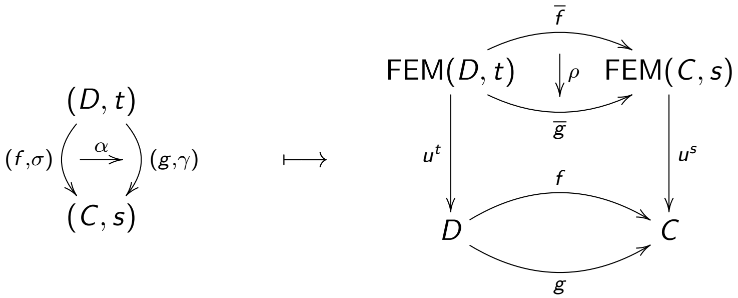
There is an embedding $I : \mathcal{D} \longrightarrow \mathsf{FEM}(\mathcal{D})$, $D \longmapsto (D, 1)$.

Explicit definition, cont.

Theorem

When a dagger 2-category $\mathcal{C}$ has FEM-objects, there is an equivalence of categories $\mathrm{FEM}(\mathcal{C}) \longrightarrow \mathcal{C}$.

Proof: By bijection of mates under the adjunction $f^t \dashv u^t$ in $\mathcal{D}$



Explicit definition, cont.

Question: Does this correspondence preserve daggers?

A 2-cell $(f, \sigma) \longrightarrow (g, \gamma)$ is a 2-cell $\alpha : f \longrightarrow gt$ in $\mathcal{D}$. It's dagger is calculated as

$$\alpha^{\dagger} t \cdot g\mu^{t\dagger} \cdot g\eta^{t} : g \longrightarrow ft$$

When $\eta^t t = t\eta^t$, the correspondence above preserves daggers

Universal property of FEM construction

Theorem

Let $\mathcal{D}$ be a dagger 2-category, and $\mathcal{C}$ a dagger 2-category with Frobenius-Eilenberg-Moore objects. Then, composition with the inclusion dagger 2-functor $I : \mathcal{D} \longrightarrow \mathsf{FEM}(\mathcal{D})$ induces an equivalence of categories

$$[\mathsf{FEM}(\mathcal{D}), \mathcal{C}]_{\mathsf{FEM}} \approx [\mathcal{D}, \mathcal{C}]$$

Dagger Frobenius wreaths again

We can construct $\mathsf{FEM}(\mathsf{FEM}(\mathcal{D}))$ for a dagger 2-category $\mathcal{D}$ – however, the induced 2-functor

$$\mathsf{wr}_{\mathcal{D}} : \mathsf{FEM}(\mathsf{FEM}(\mathcal{D})) \longrightarrow \mathsf{FEM}(\mathcal{D})$$

may not be a dagger 2-functor in general.