Weakly globular double categories and weak units

Simona Paoli[1]

[1]Department of Mathematics and Actuarial Sciences
University of Leicester (UK)

FMCS 2021

Weak 2-categories

- Classic notion: Bicategories (Bénabou)
- Various models of weak n-categories modelling examples in different areas (homotopy theory, mathematical physics, logic and computer science).
- These specialize to models of weak 2-categories, suitable equivalent to bicategories.

Segal-type models

Models of weak 2-categories based on simplicial objects $[\Delta^{op}, \mathsf{Cat}]$ satisfying extra properties:

$$\mathsf{Ta}^2 \hookrightarrow \mathsf{Ta}^2_{wg} \hookleftarrow \mathsf{Cat}^2_{wg}$$

Ta^2 Tamsamani 2-categories

$\mathsf{Cat}^2_{wg} \subset \mathsf{Cat}^2$ Weakly globular double categories

Ta^2_{wg} weakly globular Tamsamani 2-categories

Theorem [P. and Pronk, TAC 2013]: These three models are suitably equivalent. Explicit equivalence of Cat^2_{wg} with bicategories.

Kock's model

- J. Kock introduced the category Fair^2 of fair 2-categories, modelling weak 2-categories with strictly associative composition laws and weak units laws.

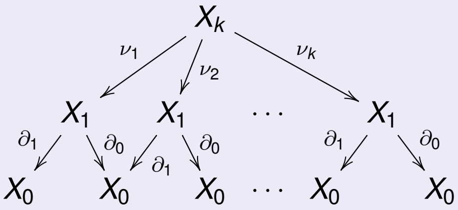
- This model is similar in flavour to the Segal-type models but is based not on the simplicial category Δ but on a different category 'fat delta' $\underline{\Delta}$.

Motivating question

- Can we directly compare Fair^2 and $\mathrm{Cat}^2_{\mathrm{wg}}$?
- We establish a direct comparison, without using the equivalences of Fair^2 and $\mathrm{Cat}^2_{\mathrm{wg}}$ with bicategories.
- This will highlight new features of weakly globular double categories and pave the way to higher dimensional generalizations (weak units conjecture).

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks.
Denote $X[k] = X_k$.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j - 1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

- There is a nerve functor

$$N : \mathrm{Cat}\ \mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\ \mathcal{C}$

$$NX \quad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow}\; X_1 \;\substack{\longrightarrow\\\longrightarrow\\\longleftarrow}\; X_0$$

Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ are isomorphisms.

Weakly globular double categories

$X \in [\Delta^{op}, \mathrm{Cat}]$ is in $\mathrm{Cat}^2_{\mathrm{wg}}$ if

i) The Segal maps are isomorphisms:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \qquad k \geq 2$$

ii) Weak globularity condition: X_0 is an equivalence relation; thus $\gamma : X_0 \to X_0^d$ is an equivalence of categories, where X_0^d is the discrete category on the set of connected components of X_0.

iii) The induced Segal maps are equivalences of categories:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \xrightarrow{\simeq} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \qquad k \geq 2$$

Coloured categories

- A coloured category is a category $\mathcal{C}$ with a subcategory $\mathcal{W}$ containing all objects. The arrows of $\mathcal{W}$ are called coloured arrows.
- Morphisms of colored categories are colour-preserving functors.
- A coloured graph is a graph in which some of the edges have been singled out as colours.
- To form the free coloured category on a coloured graph take the free category on the whole graph an let $\mathcal{W}$ be the free category on the coloured part of the graph.

Coloured ordinals

- A (finite) coloured ordinal is a free coloured category on a (finite) linearly ordered coloured graph.

- Let $\mathbb{T}$ be the category of finite non-empty coloured ordinals

Morphisms are as usual ordinals for the dots but a link can be set but may not be broken.

- Functor $\pi : \mathbb{T} \to \Delta$ contracting all the links.

Semi-categories

- Let Δ_{mono} be obtained from Δ by removing the degeneracy maps.
- If $X \in [\Delta^{op}_{mono}, \mathrm{Set}]$ satisfies the Segal condition

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \qquad k \geq 2$$

  then X is a semi-category.
- A coloured semi-category is a semi-category with a sub-semi-category comprising all objets. A morphism between coloured semi-categories is a colour preserving semi-functor.

Definition (J. Kock)

The fat delta $\underline{\Delta}$ is the category of all finite non-empty coloured semi-ordinals.

- One can naturally identify $\underline{\Delta} = \mathbb{T}_{mono}$.

- Let $(\mathrm{Cat}\,,\mathcal{W})$ be the coloured category with the arrows in $\mathcal{W}$ being the equivalences of categories.

Definition (J. Kock)

A fair 2-category is a colour-preserving functor $X : \underline{\Delta}^{op} \to \mathrm{Cat}$ preserving discrete objects and pullbacks over discrete objects.

- Denote

$$\mathcal{O} = X_{\bullet}\,, \qquad \mathcal{A} = X_{\substack{\bullet\\ \bullet}}\,, \qquad \mathcal{U} = X_{\substack{\bullet\\ |\\ \bullet}}$$

and think of these as objects, arrows, weak identity arrows.

To give a fair 2-category X it is enough to give the following data:

a) A discrete category of objects $\mathcal{O} = X_{\bullet}$, a category of arrows $\mathcal{A} = X_{\substack{\bullet \\ \bullet}}$ and a category of weak units $\mathcal{U} = X_{\substack{\bullet \\ | \\ \bullet}}$ together with a commuting diagram

$$\begin{array}{ccc} \mathcal{O} & \overset{t}{\underset{s}{\leftleftarrows}} & \mathcal{A} \\ \upuparrows & \nearrow_{u} & \\ \mathcal{U} & & \end{array}$$

b) Semi-category structures on $\mathcal{U} \rightrightarrows \mathcal{O}$ and $\mathcal{A} \rightrightarrows \mathcal{O}$ such that

$$\begin{array}{ccc} \mathcal{U} & \rightrightarrows & \mathcal{O} \\ \downarrow & & \| \\ \mathcal{A} & \rightrightarrows & \mathcal{O} \end{array}$$

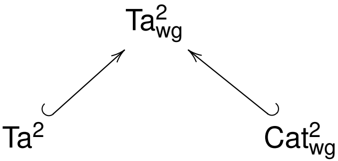
is a semi-functor.

c) The maps $\mathcal{U} \rightrightarrows \mathcal{O}$ as well as the composition maps

$$\mathcal{U} \times_{\mathcal{O}} \mathcal{A} \to \mathcal{A} \leftarrow \mathcal{A} \times_{\mathcal{O}} \mathcal{U}, \qquad \mathcal{U} \times_{\mathcal{O}} \mathcal{U} \to \mathcal{U}$$

are equivalences of categories.

Back to Segal-type models



Definition

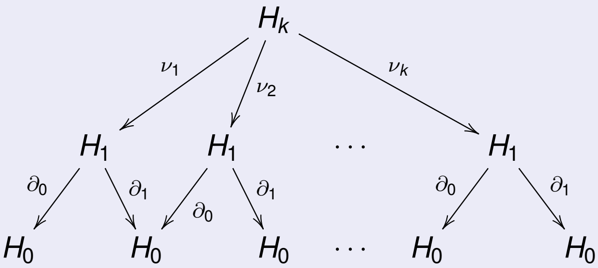
The category $\mathrm{Ta}^2_{\mathrm{wg}}$ of weakly globular Tamsamani 2-categories is the full subcategory of $[\Delta^{op}, \mathrm{Cat}\,]$ whose objects X are such that

i) X_0 is an equivalence relation.

ii) The induced Segal maps $\hat{\mu}_k : X_k \to X_k \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_k$ are equivalences of categories for all $k \geq 2$.

Segal maps for pseudo-functors

Let $H \in \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}]$ be such that H_0 is discrete. The following diagram in Cat commutes



Hence there is a unique Segal map for all $k \geq 2$

$$H_k \to H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Definition

The category $\text{SegPs}[\Delta^{op}, \text{Cat}]$ is the full subcategory of $\text{Ps}[\Delta^{op}, \text{Cat}]$ whose objects H are such that

i) H_0 is discrete.

ii) All Segal maps are isomorphisms for all $k \geq 2$

$$H_k \cong H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Theorem

There is a functor

$$Tr_2 : \mathrm{Ta}^2_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$$

$$(Tr_2 X)_k = \begin{cases} X_0^d, & k = 0 \\ X_1, & k = 1 \\ X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1, & k > 1 \,. \end{cases}$$

Further, the strictification functor $St : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}] \to [\Delta^{op}, \mathrm{Cat}]$ *restricts to a functor*

$$St : \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \to \mathrm{Cat}^2_{\mathrm{wg}} \,.$$

- The inclusion functor $i : \Delta^{op}_{mono} \to \Delta^{op}$ induces a functor $i^* : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}] \to \mathrm{Ps}[\Delta^{op}_{mono}, \mathrm{Cat}]$.

Definition

A Segalic pseudo-functor $X \in \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$ is called strong if $i^*X \in [\Delta^{op}_{mono}, \mathrm{Cat}]$. A morphism of strong Segalic pseudo-functors is a pseudo-natural transformation F in $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$ such that i^*F is a natural transformation in $[\Delta^{op}_{mono}, \mathrm{Cat}]$.

- We denote by $\mathrm{SSegPs}[\Delta^{op}, \mathrm{Cat}]$ the category of strong Segalic pseudo-functors, so that

$$i^* : \mathrm{SSegPs}[\Delta^{op}, \mathrm{Cat}] \to [\Delta^{op}_{mono}, \mathrm{Cat}] \,.$$

$\mathsf{Cat}^2_{\mathsf{wg}}$ and strong Segalic pseudo-functors

Proposition

The restriction to $\mathsf{Cat}^2_{\mathsf{wg}} \subset \mathsf{Ta}^2_{\mathsf{wg}}$ *of the functor* $Tr_2 : \mathsf{Ta}^2_{\mathsf{wg}} \to \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}\,]$ *is a functor*

$$Tr_2 : \mathsf{Cat}^2_{\mathsf{wg}} \to \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}\,].$$

- Idea of proof: To show that $i^* Tr_2 X \in [\Delta^{op}_{mono}, \mathsf{Cat}\,]$ we show that

$$\partial'_i = Tr_2 \partial_i : (Tr_2 X)_n \to (Tr_2 X)_{n-1}$$

satisfy the semi-simplicial identities $\partial'_i \partial'_j = \partial'_{j-1} \partial'_i, \quad i < j$.

From $\mathrm{Cat}^2_{\mathrm{wg}}$ to Fair^2

Theorem

There is a functor

$$F_2 : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{Fair}^2$$

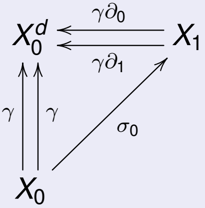
preserving 2-equivalences.

Idea of proof

- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$ define

$$(F_2X)_{\bullet} = X_0^d, \qquad (F_1X)_{\substack{\bullet\\ \bullet}} = X_1, \qquad (F_2X)_{\substack{\bullet\\ |\\ \bullet}} = X_0$$

with the commuting diagram



where $\partial_0, \partial_1 : X_1 \to X_0$ (resp. $\sigma_0 : X_0 \to X_1$) are the face (resp. degeneracy) operators in X.

Idea of proof, cont.

- Since $i^* Tr_2 X \in [\Delta^{op}_{mono}, \mathsf{Cat}]$, $i^* Tr_2 X$ is a semi-category object internal to Cat,

$$X_1 \times_{X_0^d} X_0 \longrightarrow X_1 \underset{\gamma\partial_1}{\overset{\gamma\partial_0}{\rightrightarrows}} X_0^d \ .$$

which also restricts to a semi-category structure internal to Cat

$$X_0 \times_{X_0^d} X_0 \longrightarrow X_0 \underset{\gamma}{\overset{\gamma}{\rightrightarrows}} X_0^d \ .$$

- γ as well as the following composition maps are equivalences of categories

$$X_0 \times_{X_0^d} X_0 \to X_0, \qquad X_0 \times_{X_0^d} X_1 \to X_1, \qquad X_1 \times_{X_0^d} X_0 \to X_1$$

Proposition

There is a functor

$$T_2 : \mathsf{Fair}^2 \to \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}]$$

such that, for each $X \in \mathsf{Fair}^2$, $(T_2 X)_0 = X_0$, $(T_2 X)_1 = X_1$ *and* $(T_2 X)_r = X_1 \times_{X_0} \overset{r}{\cdots} \times_{X_0} X_1$ *for* $r \geq 2$.

Definition

Let $R_2 : \mathsf{Fair}^2 \to \mathsf{Cat}^2_{\mathsf{wg}}$ be the composite

$$\mathsf{Fair}^2 \xrightarrow{T_2} \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^2_{\mathsf{wg}} \ ,$$

where St is the restriction to $\mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}]$ of the functor $St : \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}] \to \mathsf{Cat}^2_{\mathsf{wg}}$.

Theorem (P.)

The functors

$$F_2 : \mathrm{Cat}^2_{\mathrm{wg}} \rightleftarrows \mathrm{Fair}^2 : R_2$$

induce an equivalence of categories after localization with respect to the 2-equivalences

$$\mathrm{Cat}^2_{\mathrm{wg}}/\!\sim\ \simeq\ \mathrm{Fair}^2/\!\sim .$$

Summary

- Several models of weak 2-categories, in particular the Segal-type models and fair 2-categories.
- Direct comparison between weakly globular double categories and fair 2-categories.
- New light on weakly globular double categories, as encoding weak units.
- Potential for higher dimensional generalisations, leading to proof of weak units conjecture.

Reference

S. Paoli, Weakly globular double categories and weak units, arXiv:2008.11180v2

Contact e-mail address: simona.paoli@virgilio.it

Thank you for your attention!