# Linear Bicategories: Quantales and Quantaloid

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# Motivating Example: The Tropical and Arctic Semirings

These are the two semiring structures on $\mathbb{Z}^+ = \mathbb{Z} \cup \{+\infty, -\infty\}$

$$(\mathbb{Z}^+, \max, +_1) \qquad \text{and} \qquad (\mathbb{Z}^+, \min, +_2)$$

where $-\infty +_1 \infty = -\infty$ and $-\infty +_2 \infty = \infty$.

The bicategory $\mathbb{Z}^+\text{-}\mathrm{Rel}$ of sets and $\mathbb{Z}^+$-valued relations $X \overset{R}{\twoheadrightarrow} Y$ is a locally ordered linear bicategory, where $X \times Y \xrightarrow{R} \mathbb{Z}^+$.

Plan:

- Characterize quantales $Q$ such that $Q\text{-}\mathrm{Rel}$ is linear, where $Q\text{-}\mathrm{Rel}$ is the bicategory of $Q$-valued relations $X \twoheadrightarrow Y$
- Give non-locally ordered examples via Girard bicategories

# *LinearBicategories*

Introduced by Cockett, Koslowski, and Seely:

A linear bicategory $\mathcal{B}$ has two bicategory structures

$$(\otimes, \top) \quad \text{and} \quad (\oplus, \bot)$$

related via

$$A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$$

$$(A \oplus B) \otimes C \longrightarrow A \oplus (B \otimes C)$$

with naturality and coherence conditions.

# LD-quantales

Recall: A quantale $Q$ is a monoid in the category $\mathrm{Sup}$ of complete lattices and sup-preserving maps, and $Q$-$\mathrm{Rel}$ is a quantaloid, i.e., a $\mathrm{Sup}$-enriched category.

An LD-quantale is a suplattices $Q$ with operations $*, +$ and elements $\top$, $\bot$ such that

- $(Q, *, \top)$ and $(Q^{op}, +, \bot)$ are quantales
- $a * (b + c) \leq (a * b) + c$ and $(a + b) * c \leq a + (b * c)$

Example: $\mathbb{Z}^+$ with $+_1, +_2$

If $Q$ is an LD-quantale and $X \overset{R}{\longrightarrow} Y \overset{S}{\longrightarrow} Z$ in $Q$-$\mathrm{Rel}$, define

$$R \otimes S(x, z) = \sup_y (R(x, y) * S(y, z))$$

$$R \oplus S(x, z) = \inf_y (R(x, y) + S(y, z))$$

Theorem $(Q, *, +)$ *is an LD-quantale* $\iff$ *($Q$-$\mathrm{Rel}, \otimes, \oplus$) is a linear bicategory*

Proof. $(\Rightarrow)$ $R \otimes (S \oplus T) \leq (R \otimes S) \oplus T$, since

$$
\begin{aligned}
R(w,x) * \inf_y [S(x,y) + T(y,z)] &\leq R(w,x) * [S(x,y) + T(y,z)] \\
&\leq [R(w,x) * S(x,y)] + T(y,z) \\
&\leq \sup_x [R(w,x) * S(x,y)] + T(y,z)
\end{aligned}
$$

$(\Leftarrow)$ Elements $a, b, c$ of $Q$ induce $1 \overset{R_a}{\longrightarrow} 1 \overset{R_b}{\longrightarrow} 1 \overset{R_c}{\longrightarrow} 1$ in $Q$-$\mathrm{Rel}$. Since

$$R_a \otimes (R_b \oplus R_c) \leq (R_a \otimes R_b) \oplus R_c$$

it follows that $a * (b + c) \leq (a * b) + c$. □

Note: The other inequalities are similar.

# A Non-Posetal Example

$\mathcal{L}\mathrm{oc}$ locales, $(X, Y)$-bimodules $X \overset{A}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} Y$, homomorphisms

Recall (Joyal/Tierney) If $X \overset{A}{\bullet\!\!\!\to} Y \overset{B}{\bullet\!\!\!\to} Z$, then $Y \overset{A^\circ}{\bullet\!\!\!\to} X$ is a bimodule, since $A^\circ \cong \mathrm{Mod}\,Y(A, Y^\circ) \cong X\mathrm{Mod}(A, X^\circ)$, and

$$(A \otimes B)^\circ \cong \mathrm{Mod}\,Y(A, B^\circ) \cong Y\mathrm{Mod}(B, A^\circ)$$

Defining $B \oplus C = Z\mathrm{Mod}(B^\circ, C) \cong (C^\circ \otimes B^\circ)^\circ$, we get $\oplus$ is associative with left and right units $Y^\circ$ and $Z^\circ$.

Claim: $\mathcal{L}\mathrm{oc}, \otimes, \oplus$ is a linear bicategory

To define $A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$, or equivalently

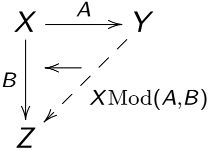
$$A \otimes Z\mathrm{Mod}(B^\circ, C) \longrightarrow Z\mathrm{Mod}(\mathrm{Mod}\,Y(A, B^\circ), C)$$

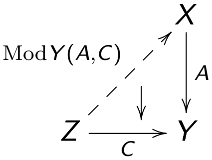
use the transpose of

$$\mathrm{Mod}\,Y(A, B^\circ) \otimes A \otimes Z\mathrm{Mod}(B^\circ, C) \xrightarrow{\varepsilon \otimes \mathrm{id}} B^\circ \otimes Z\mathrm{Mod}(B^\circ, C) \xrightarrow{\varepsilon} C$$

# Biclosed Bicategories

Recall $\mathcal{B}$ is biclosed if it has right extensions and right liftings




Note: $X\mathrm{Mod}(A,B) = A \multimap B$ and $\mathrm{Mod}Y(A,C) = C \multimapinv A$

Given $X \xrightarrow{A} Y$ and a famly $\mathcal{D} = \{ X \xrightarrow{D_X} X \,|\, X \in \mathcal{B} \}$, we get

$$A \xrightarrow{\delta_{X,A}} \mathrm{Mod}X(X\mathrm{Mod}(A, D_X), D_X)$$

and

$$A \xrightarrow{\delta_{A,Y}} Y\mathrm{Mod}(\mathrm{Mod}Y(A, D_Y), D_Y)$$

# Key Properties of $A^\circ$ in $\mathcal{L}\mathrm{oc}$

Used $A^\circ \cong \mathrm{Mod}\, Y(A, Y^\circ) \cong X\mathrm{Mod}(A, X^\circ)$

To generalize the $\mathcal{L}\mathrm{oc}$ construction, consider $\mathcal{B}$ with a family

$$\mathcal{D} = \{\, X \xrightarrow{D_X} X \mid X \in \mathcal{B}\}$$

such that

- $\delta_{X,A}$ is invertible, for all $X \xrightarrow{A} Y$ (dualizing)
- $\mathrm{Mod}\, Y(A, D_Y) \cong X\mathrm{Mod}(A, D_X)$ relative $\delta_{A,Y}, \delta_{X,A}$ (cyclic)

and define

$$A^\perp = X\mathrm{Mod}(A, D_X)$$

# Girard Bicategories

A Girard bicategory $\mathcal{B}$ is biclosed and has a cyclic dualizing family

$$\mathcal{D} = \{ X \xrightarrow{D_X} X \,|\, X \in \mathcal{B} \}$$

where $\mathcal{D}$ is called dualizing if $\delta_{X,A}$ is invertible, for all $X \xrightarrow{A} Y$; and cyclic if there are invertible cells

$$\mathrm{Mod}\,Y(A, D_Y) \cong X\mathrm{Mod}(A, D_X)$$

such that the following diagram commutes

$$\begin{array}{ccc}
 & \overset{\delta_{X,A}}{\nearrow} & \mathrm{Mod}X(X\mathrm{Mod}(A,D_X),D_X) \\
 & & \downarrow\cong \\
A & & \mathrm{Mod}X(\mathrm{Mod}Y(A,D_Y),D_X) \\
 & & \downarrow\cong \\
 & \underset{\delta_{A,Y}}{\searrow} & Y\mathrm{Mod}(\mathrm{Mod}Y(A,D_Y),D_Y)
\end{array}$$

**Lemma** $Z\mathrm{Mod}(B^{\perp}, C) \cong (C^{\perp} \otimes B^{\perp})^{\perp}$

Define $B \oplus C = Z\mathrm{Mod}(B^{\perp}, C)$. As in $\mathcal{L}\mathrm{oc}$, we get:

**Theorem** If $\mathcal{B}$ is a Girard bicategory, then $\mathcal{B}$ is a linear bicategory.

Examples:

(1) $\mathcal{Q}\mathrm{uant}$ quantales, bimodules, homomorphisms

(2) $\mathcal{Q}\mathrm{tld}$ quantaloids, profunctors, transformations

Note: $\mathcal{Q}\mathrm{uant}$ and $\mathcal{Q}\mathrm{tld}$ are bicategories of the form $\mathcal{M}\mathrm{on}(\mathcal{B})$, i.e., monads and bimodules in a bicategory $\mathcal{B}$, namely, the one object bicategory $\mathcal{S}\mathrm{up}$ and the bicategory $\mathcal{M}\mathrm{at}$ of matrices in $\mathrm{Sup}$, respectively. To establish these examples we show:

**Theorem** If $\mathcal{B}$ is a Girard bicategory with local equalizers and coequalizers stable under composition, then so is $\mathcal{M}\mathrm{on}(\mathcal{B})$.

# References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