Exponentials and Enrichment for Orbispaces

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Tangent Categories and their Applications

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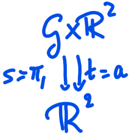
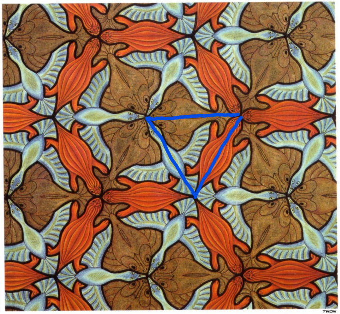
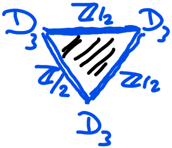
Orbispaces - Informally

- Spaces that are locally the quotient of an open subspace of Euclidean space (or another class of spaces) by the action of a finite group.
- Introduced by Satake as V-manifolds.
- Studied by Thurston and others under the name orbifolds.
- They were introduced in terms of atlases.
- We will use their representation in terms of topological groupoids.

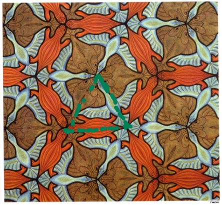
Example: A Triangular Billiard

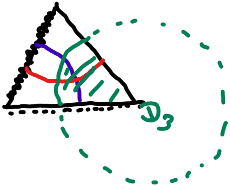


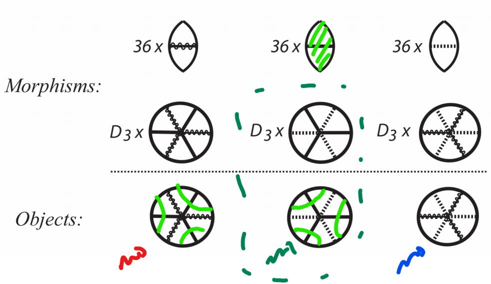




Example: A Triangular Billiard







Topological Groupoids

Definition

Orbispaces are represented by proper étale groupoids (orbigroupoids):

- $\mathcal{G}_1 \xrightarrow{(s,t)} \mathcal{G}_0 \times \mathcal{G}_0$ is proper (closed with compact fibers);
- the source map $\mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0$ is étale (and hence all structure maps are étale).

Groupoid Homomorphisms

Definition

A morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ between topological groupoid is a continuous functor; i.e., a pair of continuous maps

$$\varphi_0\colon \mathcal{G}_0 \to \mathcal{H}_0 \text{ and } \varphi_1\colon \mathcal{G}_1 \to \mathcal{H}_1$$

that makes the usual diagrams commute:

$$\begin{array}{ccccc}
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 & \xrightarrow{\mu} & \mathcal{G}_1 & \overset{t}{\underset{s}{\substack{\longrightarrow \\ \xleftarrow{u} \\ \longrightarrow}}} & \mathcal{G}_0 \\
\Big\downarrow{\scriptstyle (\varphi_1,\varphi_1)} & & \Big\downarrow{\scriptstyle \varphi_1} & & \Big\downarrow{\scriptstyle \varphi_0} \\
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 & \xrightarrow[\mu]{} & \mathcal{H}_1 & \overset{t}{\underset{s}{\substack{\longrightarrow \\ \xleftarrow{u} \\ \longrightarrow}}} & \mathcal{H}_0
\end{array}$$

However, this isn't all as far as orbispaces are concerned.

Two Presentations of a Manifold

- A manifold M can be represented by the groupoid $\mathcal{G}(M)$,

$$M \times_M M \cong M \longrightarrow M \underset{t=\mathrm{id}_M}{\overset{s=\mathrm{id}_M}{\rightrightarrows}} M \quad (u: M \to M)$$

- If $\mathcal{U}$ is an atlas for M, it can also be represented by $\mathcal{G}(\mathcal{U})$,

$$\coprod U_1 \cap U_2 \cap U_3 \longrightarrow \coprod U_1 \cap U_2 \underset{t}{\overset{s}{\rightrightarrows}} \coprod U \quad (u: \coprod U \to \coprod U_1 \cap U_2)$$

- The morphism $\mathcal{G}(\mathcal{U}) \to \mathcal{G}(M)$ is an *essential covering map*.
- There is generally no inverse for this morphism.

Essential Covering Maps

Definition

An **essential covering map** is a groupoid homomorphism of the form $\varphi_{\mathcal{U}} \colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$, where

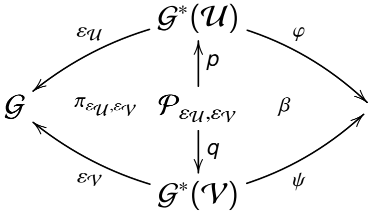
1. $\mathcal{U}$ is a locally finite collection of opens in $\mathcal{G}_0$ that meets every orbit of $\mathcal{G}$;
2. $\mathcal{G}^*(\mathcal{U})_0 = \coprod_{U \in \mathcal{U}} U$;
3. $(\varphi_{\mathcal{U}})_0$ is the inclusion embedding on each component U;
4. $\mathcal{G}^*(\mathcal{U})_1$ is the pullback

$$\begin{array}{ccc} \mathcal{G}^*(\mathcal{U})_1 & \xrightarrow{(\varphi_{\mathcal{U}})_1} & \mathcal{G}_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{G}^*(\mathcal{U})_0 \times \mathcal{G}^*(\mathcal{U})_0 & \xrightarrow[(\varphi_{\mathcal{U}})_0]{} & \mathcal{G}_0 \times \mathcal{G}_0 \end{array}$$

Orbispaces as a Bicategory of Fractions

The class $\mathfrak{C}$ of essential covering maps admits a bicalculus of right fractions and the bicategory of fractions **ProperEtaleGrpds**$(\mathfrak{C}^{-1})$ can be described by:

- **Arrows**: $\mathcal{G} \xleftarrow{\varepsilon_{\mathcal{U}}} \mathcal{G}^*(\mathcal{U}) \xrightarrow{\varphi} \mathcal{H}$, where $\varepsilon_{\mathcal{U}}$ is an essential covering map.
- **2-Cells** are diagrams of the form



where $\mathcal{P}_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}}$ is a chosen pseudo pullback square.

Hom-Categories

Remark

- The hom-categories in **ProperEtaleGpds**$(\mathfrak{C}^{-1})$ are small.
- Groupoids that are equivalent in the bicategory of fractions are called *Morita equivalent*.

What do we require from mapping objects?

When we add a topology to **Orbispaces**$(\mathcal{K},\mathcal{H})$ to make it a topological groupoid $[\mathcal{K},\mathcal{H}]$ we may aim for either of the following:

1. **Make $\mathcal{K}$ Exponentiable:**

$$\textbf{OrbiGrpds}(\mathcal{G}\times\mathcal{K},\mathcal{H}) \simeq \textbf{OrbiGrpds}(\mathcal{G},[\mathcal{K},\mathcal{H}]).$$

   Note: this determines $[\mathcal{K},\mathcal{H}]$ up to Morita equivalence if it exists.

2. **Bicategorical Enrichment:** composition defines a morphism

$$[\mathcal{G},\mathcal{K}]\times[\mathcal{K},\mathcal{H}] \to [\mathcal{G},\mathcal{H}]$$

   that satisfies the usual conditions up to coherent natural isomorphisms.
   Note: this has more than one solution.

What has been done in the literature?

- Weimin Chen, On a notion of maps between orbifolds, I. Function Spaces, *Communications in Contemporary Mathematics* **8** no.5 (2006), 569-620
- David Carchedi, Compactly generated stacks: a Cartesian closed theory of topological stacks, *Advances* **229** (2009)
- Behrang Noohi, Mapping stacks of topological stacks, *Journal für die reine und angewandte Mathematik* **646** (2010)

Topological Groupoids

For the 2-category of **TopGrpds** of groupoids in locally compact spaces there are topological mapping groupoids **GMap**$(\mathcal{G},\mathcal{H})$ such that

- There is an isomorphism

$$\textbf{TopGrpds}(\mathcal{G}\times\mathcal{K},\mathcal{H})\simeq\textbf{TopGrpds}(\mathcal{G},\textbf{GMap}(\mathcal{K},\mathcal{H}))$$

- Composition defines a continuous functor

$$\textbf{GMap}(\mathcal{G},\mathcal{K})\times\textbf{GMap}(\mathcal{K},\mathcal{H})\to\textbf{GMap}(\mathcal{G},\mathcal{H}).$$

Topology for the Exponential: the Space of Objects

- [Bustillo-P-Szyld] Homs in a bicategory of fractions are 2-filtered pseudo colimits of homs in the original bicategory.
- So we take $\mathbf{OMap}(\mathcal{K},\mathcal{G})$ to be the *pseudo colimit of the topological groupoids* $\mathbf{GMap}(\mathcal{K}^*(\mathcal{U}),\mathcal{G})$.
- Assume that $\mathcal{K}$ is *orbit-compact*: $\mathcal{K}_0/\mathcal{K}_1$ is compact.
- Topology on the space of objects:

$$\mathbf{OMap}(\mathcal{K},\mathcal{G})_0 = \coprod_{\varepsilon_{\mathcal{U}}} \mathbf{GMap}(\mathcal{K}^*(\mathcal{U}),\mathcal{G})_0$$

  where the coproduct is taken over all essential covering maps $\varepsilon_{\mathcal{U}}\colon \mathcal{K}^*(\mathcal{U}) \to \mathcal{K}$ with $\mathcal{U}$ finite essential coverings of $\mathcal{K}_0$.
- Note: $\mathbf{GMap}(\mathcal{K}^*(\mathcal{U}),\mathcal{G})_0 \subseteq \mathbf{Map}(\mathcal{K}^*(\mathcal{U})_1,\mathcal{G}_1)$.

Topology for the Exponential: the Space of Arrows

- Points in this space are diagrams

$$\begin{array}{ccccc} & & \mathcal{K}^*(\mathcal{U}) & & \\ & \swarrow^{\varepsilon_{\mathcal{U}}} & \uparrow \pi_{\mathcal{U}} & \searrow^{\varphi} & \\ \mathcal{K} & \pi_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}}\Downarrow & \mathcal{P}_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}} & \beta\Downarrow & \mathcal{G} \\ & \nwarrow_{\varepsilon_{\mathcal{V}}} & \downarrow \pi_{\mathcal{V}} & \nearrow_{\psi} & \\ & & \mathcal{K}^*(\mathcal{V}) & & \end{array}$$

- Then

$$\mathbf{OMap}(\mathcal{K},\mathcal{G})_1 = \coprod_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}} \mathcal{Q}_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}}.$$

- Where

$$\begin{aligned} \mathcal{Q}_{\varepsilon_{\mathcal{U}}.\varepsilon_{\mathcal{V}}} = \quad & \mathbf{GMap}(\mathcal{K}^*(\mathcal{U}),\mathcal{G})_0 \\ & \times_{(\pi_{\mathcal{U}}^*,\mathbf{GMap}(\mathcal{P}_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}},\mathcal{G})_0,s)} \quad \mathbf{GMap}(\mathcal{P}_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}},\mathcal{G})_1 \\ & \times_{(t,\mathbf{GMap}(\mathcal{P}_{\varepsilon_{\mathcal{U}},\varepsilon_{\mathcal{V}}},\mathcal{G})_0,\pi_{\mathcal{V}}^*)} \quad \mathbf{GMap}(\mathcal{K}^*(\mathcal{V}),\mathcal{G})_0 \end{aligned}$$

OMap$(\mathcal{G}, \mathcal{H})$ is the Exponential

- **OMap**$(\mathcal{K}, \mathcal{G})$ is proper and fits in an equivalence of categories

$$\textbf{Orbigpds}(\mathfrak{C}^{-1})(\mathcal{H} \times \mathcal{K}, \mathcal{G}) \simeq \textbf{Orbigpds}(\mathfrak{C}^{-1})(\mathcal{H}, \textbf{OMap}(\mathcal{K}, \mathcal{G}))$$

  when $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{G}$ are proper étale groupoids.
- **OMap**$(\mathcal{K}, \mathcal{G})$ is generally not étale.
- We will find a smaller Morita equivalent groupoid **AMap**$(\mathcal{K}, \mathcal{G})$ that has all the desired properties.

Admissible Generalized Morphisms

A span

$$\mathcal{K} \xleftarrow{\varepsilon} \mathcal{K}^*(\mathcal{U}) \xrightarrow{\varphi} \mathcal{G}$$

is called **admissible** if

- for each $U \in \mathcal{U}$, $\varepsilon(U) \subseteq \mathcal{G}_0$ is relatively compact: its closure is compact;
- φ can be extended to these closures.

AMap$(\mathcal{K},\mathcal{G})_0$

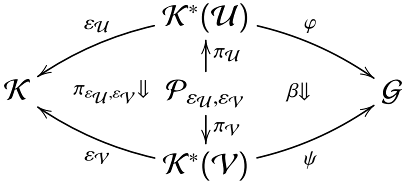
- Write **AMap**$(\mathcal{K},\mathcal{G})$ for the full subcategory of **OMap**$(\mathcal{K},\mathcal{G})$ on admissible spans with finite essential coverings, but we don't use the subspace topology!
- Topology on the space of objects:

$$\mathbf{AMap}(\mathcal{K},\mathcal{G})_0 = \coprod_{\varepsilon_{\mathcal{U}}} \mathbf{GMap}(\mathcal{K}^*(\overline{\mathcal{U}}),\mathcal{G})_0$$

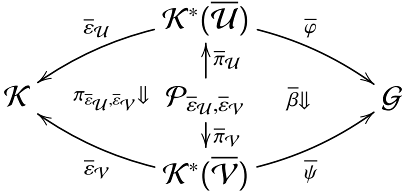
  where the coproduct is taken over all $\varepsilon_{\mathcal{U}} \colon \mathcal{K}^*(\mathcal{U}) \to \mathcal{K}$ with $\mathcal{U}$ a finite essential covering of $\mathcal{K}_0$ by relatively compact opens.
- Here, $\mathcal{K}^*(\overline{\mathcal{U}})_0 = \coprod_{U\in\mathcal{U}} \overline{U}$, with the closures taken in $\mathcal{K}_0$, and $\mathcal{K}^*(\overline{\mathcal{U}})_1$ is defined by pullback.

AMap$(\mathcal{K},\mathcal{G})_1$

- Points in **AMap**$(\mathcal{K},\mathcal{G})_1$ are diagrams of admissible spans



and can be topologized using the closures and unique extensions:



Properties of the Exponential

Remark

Shrinkage gives rise to an essential equivalence $\mathbf{OMap}(\mathcal{K},\mathcal{G}) \to \mathbf{AMap}(\mathcal{K},\mathcal{G})$.

Theorem

Let $\mathcal{G}$, $\mathcal{L}$ and $\mathcal{G}$ be paracompact, locally compact, proper, étale groupoids.

- *When $\mathcal{K}$ is orbit compact,*

$$\mathbf{Orbispaces}(\mathcal{L}\times\mathcal{K},\mathcal{G}) \simeq \mathbf{Orbispaces}(\mathcal{L},\mathbf{AMap}(\mathcal{K},\mathcal{G})).$$

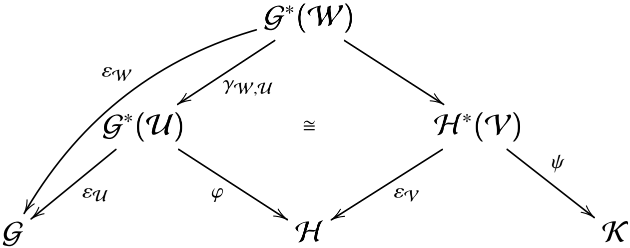
- *The groupoid $\mathbf{AMap}(\mathcal{K},\mathcal{G})$ is proper and étale.*
- *If $\mathcal{K}$ and $\mathcal{K}'$ are Morita equivalent and $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent, then $\mathbf{AMap}(\mathcal{K},\mathcal{G})$ and $\mathbf{AMap}(\mathcal{K}',\mathcal{G}')$ are Morita equivalent.*

Enrichment?

- The composition functor

$$\mathbf{AMap}(\mathcal{G},\mathcal{H}) \times \mathbf{AMap}(\mathcal{H},\mathcal{K}) \to \mathbf{AMap}(\mathcal{G},\mathcal{K})$$

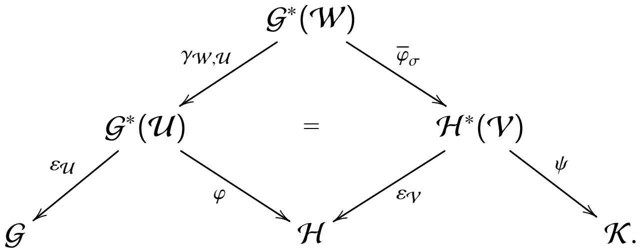
  is not continuous in general.



- Solution: construct a *generalized map* in its place: introduce a covering so that on each part we can use a fixed $\mathcal{W}$.
- Then we enrich over orbispaces rather than topological groupoids.
- Note: this works only for orbit-compact orbifolds.

Composition

Let $\gamma\colon \mathcal{G}^*(\mathcal{W}) \to \mathcal{G}^*(\mathcal{U})$ and $\sigma\colon \mathcal{W} \to \mathcal{V}$. For $(\varphi, \psi) \in \mathcal{O}[\gamma_{\mathcal{W},\mathcal{U}}; \sigma] = \{(\varphi, \psi);\ \varphi(\overline{\gamma_{\mathcal{W},\mathcal{U}}(W)}) \subseteq \varepsilon_{\mathcal{V}}\sigma(W) \text{ for all } W \in \mathcal{W}\}$, composition is defined as follows:



where $\overline{\varphi}_\sigma$ is the unique arrow that makes the square commute.

Conclusions

Assume that all orbigroupoids here are étale, proper, paracompact and locally compact.

- For orbit-compact $\mathcal{K}$, **AMap**$(\mathcal{K}, \mathcal{G})$ is the exponential in the bicategory of orbispaces.
- For orbit-compact $\mathcal{K}$, **AMap**$(\mathcal{K}, \mathcal{G})$ is étale and proper.
- The subbicategory on orbit-compact, paracompact, locally compact orbispaces is enriched over the bicategory of proper étale groupoids with generalized maps with the **AMap**$(\mathcal{K}, \mathcal{G})$ as the mapping objects.

Example 1

- If $\mathcal{G}$ and $\mathcal{H}$ are orbigroupoids with only trivial isotropy groups and $X = \mathcal{G}_0/\mathcal{G}_1$ and $Y = \mathcal{H}_0/\mathcal{H}_1$ are the underlying spaces, then **AMap**$(\mathcal{G}, \mathcal{H})$ is orbispace equivalent to **Map**(X, Y), the ordinary mapping space for topological spaces.
- If $\mathcal{G}$ and $\mathcal{H}$ are finite discrete groups,

$$\mathbf{AMap}(\star_{\mathcal{G}}, \star_{\mathcal{H}}) = \mathbf{GMap}(\star_{\mathcal{G}}, \star_{\mathcal{H}})$$

  with space of objects **GrpHom**$(\mathcal{G}, \mathcal{H})$ with the discrete topology and space of arrows the discrete space with

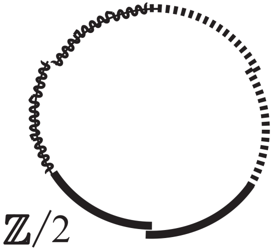
$$(s,t)^{-1}(\varphi, \psi) = \{h \in \mathcal{H};\ \psi = h\varphi h^{-1}\}.$$

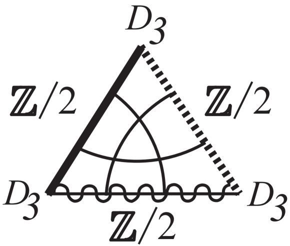
Example 2: the Triangular Billiard

Let $\mathcal{T}$ be the triangular billiard orbispace and $\mathcal{P}_2 = \star_{\mathbb{Z}/2}$ a point with (trivial) $\mathbb{Z}/2$ action. Then

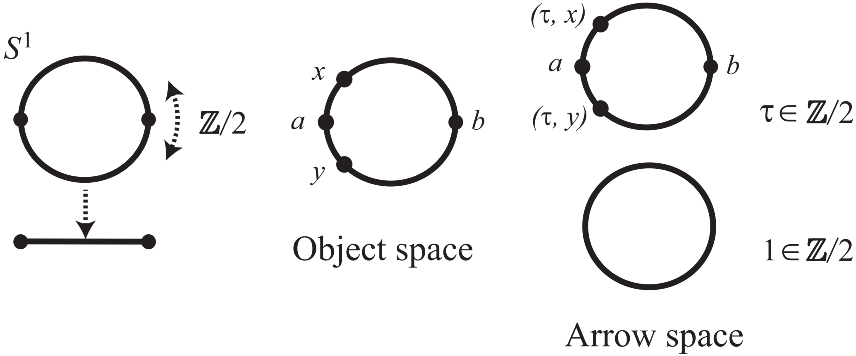
$$\mathbf{AMap}(\mathcal{P}_2, \mathcal{T}) \simeq \mathcal{T} \coprod \mathcal{S}_2$$

where $\mathcal{S}_2$ is a circle S^1 with trivial $\mathbb{Z}/2$ action.





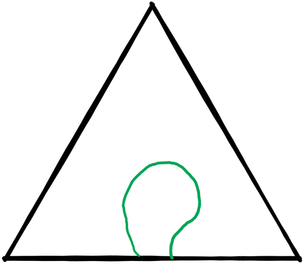
S^1 with a $\mathbb{Z}/2$-Action



This orbifold is also called the interval with silvered endpoints.

Example 3: Paths with Silvered Endpoints

Let $\mathcal{I}_{2,1,2}$ be the interval with two silvered endpoints and $\mathcal{T}$ the triangular billiard orbispace. The mapping space $\mathbf{AMap}(\mathcal{I}_{2,1,2}, \mathcal{T})$ of "silvered paths" has only two connected components: the component of ordinary paths and the component of this path



Thank you for your attention!