Tangent Infinity Categories

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connecting: BJORT*[1] and recent work with Matthew Burke & Michael Ching

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[1] Brenda Johnson, Christina Osborne, Emily Riehl and Amelia Tebbe

Outline

The goal of this talk is to explain the connection between categorical differentiation and functor calculus. To do so, we need to invent a new kind of "homotopical" tangent category. A rough outline of the talk is as follows.

1. Functor Calculus
2. Weil-algebras
3. Tangent infinity categories and examples
4. The Goodwillie tangent structure
5. Differential objects and differentiation
6. n-jets

Functor Calculus (Goodwillie)

Let $F : \mathcal{C} \to \mathcal{D}$ be a homotopy functor of model categories.

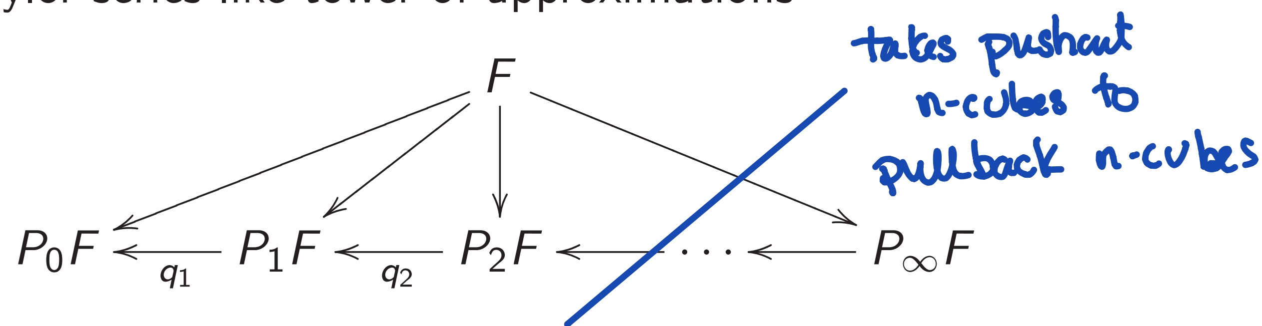
- $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are abelian; (classical homological algebra)
- $F : Top \to Top$ of pointed topological spaces (homotopy theory).

Definition (Excisive)

A functor F is excisive if it takes homotopy pushouts to homotopy pullbacks, e.g.

$$F(X \vee Y) = F(X) \times F(Y).$$

There is a Taylor series-like tower of approximations



where $P_n F$ is the best *n*-excisive approximation to F, and $D_n F = \operatorname{hofib} q_n$ is homogeneous *n*-excisive. The functor $D_1 F$ is excisive and reduced.

Functor calculus & Cartesian differential categories (Johnson, Lemay)

Theorem (B.-Johnson-Osborne-Riehl-Tebbe)

The (homotopy) category of abelian categories is a Cartesian differential category.

The derivative of a functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories is

$$\nabla F(V, A) = D_1 F(A \oplus -)(V)$$

Theorem (Blute-Cockett-Seely '09 & Cockett-Lemay '20)

A category has differentiation iff it has linearization.

The linearization of a map in a cartesian left additive category is *additive*. Since *Top* doesn't have biproducts, excisive $\neq$ additive. In particular:

$$D_1 F(X \vee Y) = D_1 F(X) \times D_1 F(Y).$$

Weil modules (Garner, I)

Tangent categories are hard to define 'up to homotopy' (a nightmare of coherence conditions!), but there is another perspective due to Leung.

Definition

There is a symmetric monoidal category $\mathbb{W}eil$ with

- Objects are augmented commutative $\mathbb{N}$-algebras of the form

$$A = \mathbb{N}[x_1, \ldots, x_n]/(x_i x_j, i \simeq j)$$

- Morphisms are maps of augmented commutative $\mathbb{N}$-algebras,
- $\otimes$ is the monoidal product.

Let W^n denote the n-fold product of W with itself. Every object in $\mathbb{W}eil$ is of the form $W^{n_1} \otimes \cdots \otimes W^{n_r}$.

The tangent pullbacks (Garner, I)

There are pullbacks:

$$
\begin{array}{ccc}
W^2 & \xrightarrow{\mu} & W \otimes W \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle 1 \otimes \epsilon} \\
\mathbb{N} & \xrightarrow{\eta} & W
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \otimes W^{m+n} & \longrightarrow & A \otimes W^{m} \\
\downarrow & & \downarrow \\
A \otimes W^{n} & \longrightarrow & A
\end{array}
$$

Here, $\mu(x) = ab$, $\mu(y) = b$, η is the unit and ϵ is the augmentation.

The first of these corresponds to the ‘universality of the vertical lift’ in Cockett-Cruttwell. The second corresponds to the requirement that the tangent bundle functor must preserve products.

$\mathbb{W}$*eil* and tangent categories (Garner, I)

Theorem (Leung)

A category $\mathcal{X}$ is a tangent category iff there is a strong monoidal functor

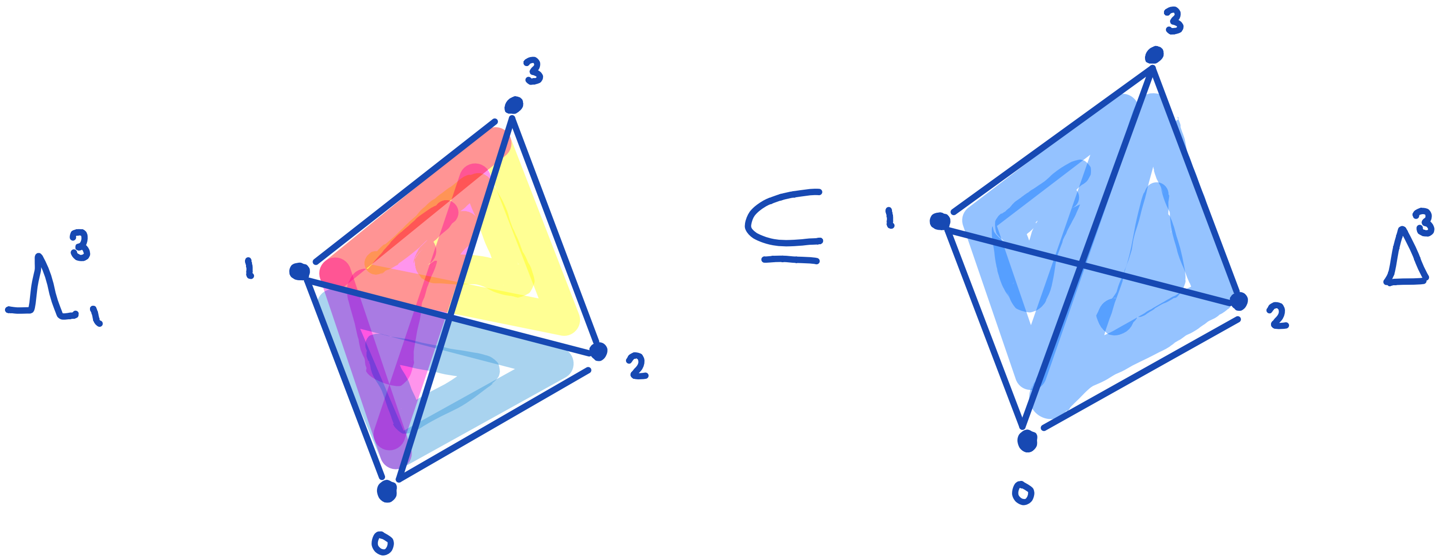
$$T : (\mathbb{W}eil, \otimes) \to (\mathrm{End}(\mathcal{X}), \circ)$$

which preserves the tangent pullbacks.

We apply this theorem as definition for infinity categories rather than ordinary categories.

Simplicial sets

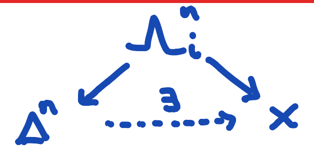
A simplicial set is a functor $\Delta^{op} \to \mathit{Set}$, pictured geometrically as a topological space:



A tent made of 3 triangular flaps glued along common edges

A solid tetrahedron.

Every Λ^n_i you find in X can be filled in.

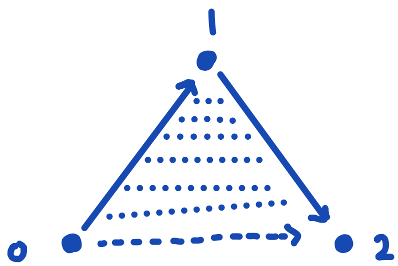


I mean...

Definition (Infinity Categories (Boardman-Vogt, Joyal))

An infinity category (quasi-category) is a simplicial set in which every inner horn has a filler.

Every category is an infinity category. In particular, $\mathbb{W}\mathit{eil}$ is an infinity category.



Given $0 \to 1 \to 2$ in $\mathscr{C}$
There is indeed a unique arrow $0 \to 2$ in $\mathscr{C}$ determined by composing.
This fills the horn!

An ∞-functor is just a map of simplicial sets.

Fun with infinity categories

Many of the things you can do with ordinary categories can be done with quasi-categories:

- There is a function complex $Fun(\mathbb{X}, \mathbb{Y})$ which is again a quasi-category.
- A monoidal ∞-category is a simplicial monoid $M^{\otimes}$ for which the underlying simplicial set is a quasi-category.
- A strict monoidal ∞-functor is a map of simplicial sets which preserves the monoidal structure.
- A strong monoidal ∞-functor is a map of simplicial sets which preserves the monoidal structure *up to coherent isomorphism (homotopy)*.

e.g. The things that behave properly with (homotopy) colims.

The category $\mathbb{W}eil$ is a monoidal infinity category, and it is cofibrant as a monoidal infinity category.

Definition

A tangent infinity category is an infinity category $\mathbb{X}$ together with a *strict* monoidal functor

$$T : \mathbb{W}eil^{\otimes} \to End(\mathbb{X})^{o}$$

for which the underlying map of quasi-categories preserves the tangent pullbacks.

Examples:

- Any tangent category is a tangent ∞-category.
- An arbitrary infinity category $\mathbb{X}$ has a trivial tangent structure given by $T(A) = Id_{\mathbb{X}}$.

The Goodwillie tangent structure

An infinity category $\mathcal{C}$ is Lurie-differentiable if it admits finite limits and sequential colimits, and those commute. Let $\mathbb{C}at_\infty$ be the ∞-category of ∞-categories, and $\mathbb{C}at_\infty^{diff} \subset \mathbb{C}at_\infty$ the subcategory whose objects are Lurie-differentiable ∞-categories and whose morphisms are functors that preserve sequential colimits.

Finite simplicial sets

We say that a s. set is finite if it is homotopy equivalent to the singular s. set of a finite CW complex. Let $\mathcal{S}_{fin,*}$ denote the simplicial nerve of the simplicial category in which an object is a pointed finite Kan complex, with enrichment given by the pointed mapping spaces. Since the mapping spaces are Kan-complexes, $\mathcal{S}_{fin,*}$ is a quasi-category.

Lurie defines the tangent bundle on a Lurie-differentiable ∞-category $\mathcal{C}$ to be the ∞-category

$$T(\mathcal{C}) := Exc(\mathcal{S}_{fin,*}, \mathcal{C})$$

of excisive functors.

The Goodwillie tangent structure

Theorem (B-Burke-Ching)

The tangent bundle can be extended to a functor $T : \mathbb{W}eil^{\otimes} \to End(\mathbb{C}at_{\infty}^{diff})^{o}$ giving a tangent infinity structure on $\mathbb{C}at_{\infty}^{diff}$.

Some hints about the proof:

- $p : T^{W}(\mathcal{C}) \to \mathcal{C}$ given by $L \mapsto L(*)$
- If $A = W^{n_1} \otimes \cdots \otimes W^{n_r}$, then $T^{A}(\mathcal{C}) = Exc^{1,\ldots,1}(\mathcal{S}^{n_1}_{fin,*} \times \cdots \times \mathcal{S}^{n_r}_{fin,*}, \mathcal{C})$, i.e. functors which are excisive in each of r variables separately.
- If $F : \mathcal{C} \to \mathcal{D}$, then $T^{A}(F) := P_{A}(F_{*})$, i.e. the excisive approximation to post-composition with F. (Note: $P_1(FL) = P_1(F(P_1 L))$.)
- If $\phi : A \to A'$, then $T^{\phi}(\mathcal{C}) : T^{A}(\mathcal{C}) \to T^{A'}(\mathcal{C})$ mirrors the map ϕ by treating factors of $\mathcal{S}^{n}_{fin,*}$ like the factorization of A into W's.

Stable infinity categories

An infinity category $\mathcal{C}$ is stable if it is pointed, admits finite limits and colimits, and a commuting square diagram is a pushout iff it is a pullback.

Theorem (B-Burke-Ching)

A Lurie-differentiable ∞-category $\mathcal{C}$ is differential (in the sense of Cockett-Cruttwell) iff $\mathcal{C}$ is a stable ∞-category.

The proof follows from two results due to Lurie. If $\mathcal{C}$ is any ∞-category and $X \in \mathcal{C}$, then

$$T_X\mathcal{C} \simeq Exc_*(\mathcal{S}_{fin,*}, \mathcal{C}/X)$$

of *reduced* excisive functors, which Lurie proved is stable.
On the other hand, if $\mathcal{C}$ is stable, then

$$T_*\mathcal{C} = Exc_*(\mathcal{S}_{fin,*}, \mathcal{C})$$

which is equivalent to $\mathcal{C}$ by a result of Lurie. That is, $\mathcal{C}$ is equivalent to a tangent space.

Connection to BJORT (Johnson)

By work of Cockett-Cruttwell, the differential objects of a tangent category are a Cartesian differential category. The homotopy category of stable, Lurie-differentiable infinity categories is a Cartesian differential category. The derivative of a functor F of stable infinity categories is

$$\nabla(F)(V, X) := D_1(F(X \oplus -))(V)$$

exactly as in the BJORT case.

This is not surprising, because we tend to think of homological algebra as an instance of a stable infinity category (e.g. the derived category of an abelian category is a stable infinity category). But the category used in BJORT itself is not exactly of this type.

Connection to n-excisive functors

In the category of smooth manifolds, two smooth functions $f, g : M \to N$ have Taylor series at x (multivariate, local coordinates) that agree to degree n iff $T^n_x(f) = T^n_x(g)$. The equivalence class of f under this relation is the n-jet of f. The n-jets determine the degree n Taylor polynomial (and vice-versa).

A functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathbb{C}at^{diff}_\infty$ can be thought of locally by restricting to the slice category, $F/X : \mathcal{C}/X \to \mathcal{D}$. Let $P^X_n F$ denote $P_n(F/X) : \mathcal{C}/X \to \mathcal{D}$.

Likewise, the n-fold tangent space $T^n_X\mathcal{C}$ of $\mathcal{C}$ at X is the fiber of $T^n\mathcal{C} \to \mathcal{C}$ over X. Let $\iota_X : T^n_X\mathcal{C} \to T^n\mathcal{C}$ be the inclusion of the fiber.

If F is reduced, take $P^*_n F = P_n F$ (ignore slice)

Connection to n-excisive functors

if F, G reduced and $X = *$...

Theorem

Analogue of n-jets Let $F, G : \mathcal{C} \to \mathcal{D}$ in $\mathbb{C}\mathrm{at}_\infty^{diff}$ and $\alpha : F \Rightarrow G$, and let $F/X : \mathcal{C}/X \to \mathcal{D}$ denote the restriction of F to the slice category. Then

$$P_n^X(\alpha) : P_n^X(F) \Rightarrow P_n^X(G)$$

$P_n F \simeq P_n G$ iff

is an equivalence if and only if

$$T_X^n \alpha \iota_X : T_X^n F \iota_X \Rightarrow T_X^n G \iota_X$$

$T_*^n F i_* \simeq T_*^n G i_*$ i.e. fibers are $\simeq$

is an equivalence.

Upshot: The n-jet of F is $[P_n F]$, and this says you may as well use $T_*^n F$ instead.

THANK YOU!!

Bauer, Burke, Ching, *Tangent Infinity Categories*, https://arxiv.org/pdf/2101.07819.pdf