Endoscopy and Stabilization for Symmetric varieties

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Basic Functions, Orbital Integrals, and Beyond Endoscopy

Outline

1. First case: twisted base change
2. Unitary FJ periods and Xiao–Zhang RTF
3. Endoscopic symmetric spaces
4. Local results for unitary FJ periods

Twisted base change

Let F be a number field and let G be a classical group (eg: $\mathrm{Sp}_{2n}, U_n, \mathrm{O}_n$) with $\mathrm{H} \subset \mathrm{G}$ a (spherical) subgroup.
Getz-Wambach: Let $\theta^2 = 1$ be such that $\mathrm{GL}_N^\theta = \mathrm{G}$, and let $\sigma^2 = 1$ be an involution on G .

- Set $\mathrm{H} = \mathrm{G}^\sigma$.
- $\tau = \sigma \circ \theta$ is involution on GL_N. Set $\mathrm{H}_1 = \mathrm{GL}_N^\sigma$ and $\mathrm{H}_2 = \mathrm{GL}_N^\tau$.

General Principle (coarse form)

Let π be a cuspidal rep of $\mathrm{G}(\mathbb{A})$ and let Π be its functorial lift to $\mathrm{GL}_N(\mathbb{A})$. TFAE

1. Π is H_1 and H_2-distinguished
2. there exists π' nearly equivalent to π which is H-distinguished

Relative twisted endoscopy

Strategy: compare the **stabilized relative trace formula** of G encoding H-periods with the appropriate twisted relative trace formula for GL_N

Example (biquadratic case, Getz-Wambach)

Let E/F and M/F be two quadratic extensions such that ME/F is biquadratic. Consider $\mathrm{H} = U_n$ associated to M/F and $\mathrm{G} = \mathrm{Res}_{E/F}(\mathrm{H}_E)$. Let σ such that $\mathrm{G}^\sigma = \mathrm{H}$. Then if $\mathrm{G}' = \mathrm{GL}_{n,ME}$,

$$\mathrm{H}_1 = \mathrm{GL}_{n,M} \qquad \text{and} \qquad \mathrm{H}_2 = U_{n,L} \qquad (L = (ME)^\theta).$$

Then relates H-periods on G to Asai L-functions for base change to ME/F.

- Rely on simple trace formula to avoid instability issues

Unitary Friedberg–Jacquet periods

Let E/F be quadratic extension of number fields, and take $\mathrm{G} = U_{2n}$. Let σ fix $\mathrm{H} = U_n \times U_n$. Then if $\mathrm{G}' = \mathrm{GL}_{2n,E}$,

$$\mathrm{H}_1 = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E} \quad \text{and} \quad \mathrm{H}_2 = U'_{2n} \quad \text{(an inner form of G)}$$

- H_1-dist. cuts out symplectic type (i.e. $L(s, \pi, \wedge^2)$ has a pole at $s = 1$),
- H_2-dist. cuts out the image of base change from $\mathrm{GL}_{2n,F}$,

Conjecture

Let π be tempered cuspidal on $\mathrm{GL}_{2n}(\mathbb{A}_F)$ of symplectic type and $\Pi = BC(\pi)$. TFAE:

1. $L(\frac{1}{2}, \Pi) \neq 0$,
2. *there exists an (G, H) and cusp form π_G on $\mathrm{G}(\mathbb{A}_F)$ such that $\Pi = BC(\pi_G)$ and π_G is H-distinguished.*

Partial Results: Pollack–Wan–Zydor ('19), Chen–Gan ('21)

Relative trace formulas

Xiao-Zhang relative trace formula

Propose a RTF

$$I(f', \Phi, s) := \iint^{*}_{[\mathrm{GL}_n \times \mathrm{GL}_n]^2_F} K_{f'}(x, y)\eta_{E/F}(x)|xy|^s E_{\Phi}(y)dxdy$$

designed to detect symplectic-type cusp forms on $\mathrm{GL}_{2n,F}$, encoding base change L-function $L(s+\frac{1}{2}, \Pi)$

Set $\mathrm{X} = U_{2n}/U_n \times U_n$ and consider the distribution

$$TF_{\mathrm{X}}(f) := \int^{*}_{[\mathrm{H}]} \left(\sum_{x \in \mathrm{X}(F)} f(h^{-1}x) \right) dh = \sum_x \mathrm{vol}([\mathrm{H}_x])\mathrm{O}_x(f) + \dots$$

with $f \in C_c^{\infty}(\mathrm{X}(\mathbb{A}_F))$ and x ranges over $\mathrm{H}(F)$-orbits on $\mathrm{X}^{ell}(F)$.

Stable comparison

We have a matching of stable regular semi-simple orbits

$$\mathrm{GL}_n(\overline{F})^2 \backslash \mathrm{GL}_{2n}(\overline{F}) \times \overline{F}^n / \mathrm{GL}_n(\overline{F})^2$$
$$\downarrow$$
$$\mathrm{H}(\overline{F}) \backslash \mathrm{X}(\overline{F})$$

- This does not descend to orbits over F.
- **Conjecture:** $\exists\, f \leftrightarrow f' \otimes \Phi$ with matching (stable) orbital integrals at $s = 0$.
- J.Xiao-W. Zhang (in progress): prove fundamental lemma for unit.

Need to stabilize $TF_{\mathrm{X}}(f)$ to compare with $I(f', \Phi, 0)$.

Stable comparison and Prestabilization

Using techniques of Labesse, we first prestabilize:

Proposition (L)

The elliptic part of the RTF admits a prestabilization:

$$TF_{\mathrm{X}}^{ell}(f) = STF_{\mathrm{X}}^{ell}(f) + \sum_{[x]} \sum_{\kappa \neq 1} \prod_{v} \mathrm{O}_{x}^{\kappa}(f_v)$$

where

- $[x]$ is over stable H-orbits on $\mathrm{X}^{ell}(F)$ (relative elliptic locus)
- $\kappa \in H^1(\mathbb{A}_F, \mathrm{H}_x)^D$ ranges over certain characters.

The **stable part**

$$STF_{\mathrm{X}}^{ell}(f) = \sum_{[x]} \prod_{v} \mathrm{O}_{x}^{st}(f_v)$$

may be compared to Xiao-Zhang's RTF.

Stabilization of the elliptic part

For each $a+b=n$, we introduce the **endoscopic symmetric variety**

$$\mathrm{X}_{a,b} = (U_{2a}/U_a \times U_a) \times (U_{2b}/U_b \times U_b).$$

Theorem (L, '19; '20; in preparation)

For "regular" $f \in C_c^\infty(\mathrm{X}(\mathbb{A}_F))$,

$$TF_{\mathrm{X}}^{ell}(f) = \sum_{n=a+b} \iota(a,b) STF_{a,b}^{\mathrm{X}-ell}(f^{a,b}).$$

with $f^{a,b} \in C_c^\infty(X_{a,b}(\mathbb{A}))$.

- analogous to stabilization of elliptic part of Arthur-Selberg trace formula (Langlands, Shelstad, Kottwitz, Waldspurger, Ngô,...)
- Each of the endoscopic terms may be compared with lower-rank versions of Xiao–Zhang's RTF

κ-Orbital Integrals

- Now let F be a local field of characteristic 0
- For $x \in \mathrm{X}^{rss}(F)$ and $f \in C_c^\infty(\mathrm{X}(F))$, the κ-orbital integral is

$$\mathrm{O}_x^\kappa(f) = \sum_{x'} \langle \mathrm{inv}(x,x'), \kappa\rangle \mathrm{O}_{x'}(f)$$

where $inv(x,x') \in H^1(F, \mathrm{H}_x)$ and $\kappa \in H^1(F, \mathrm{H}_x)^D$. Here,

$$\mathrm{O}_x(f) = \int_{\mathrm{H}_x(F)\backslash \mathrm{H}(F)} f(h^{-1}\cdot x)dh.$$

Endoscopic symmetric variety

I now want to explain how the character κ determines an associated endoscopic variety

$$\mathrm{X}_{a,b} = (U_{2a}/U_a \times U_a) \times (U_{2b}/U_b \times U_b).$$

Dual group of X (Nadler–Gaitsgory, Knop–Schalke)

If $\mathrm{X} = \mathrm{G}/\mathrm{H}$ is a symmetric variety*, then there is a dual group $\check{\mathrm{G}}_{\mathrm{X}}$, equipped with a morphism φ_{X} such that

- if $T \subset \mathrm{G}$ is a maximal torus generically acting through the quotient $T \to A_{\mathrm{X}}$,

$$\begin{array}{ccc} \check{A}_{\mathrm{X}} & \longrightarrow & \check{T} \\ \downarrow & & \downarrow \\ \check{\mathrm{G}}_{\mathrm{X}} & \xrightarrow{\varphi_{\mathrm{X}}} & \check{\mathrm{G}} \end{array}$$

  commutes. In our case, $\check{\mathrm{G}} = \mathrm{GL}_{2n}$ and $\check{\mathrm{G}}_{\mathrm{X}} = \mathrm{Sp}_{2n}$.
- In our setting, there is a complex **dual symmetric variety** (Nadler)

$$\check{\mathrm{X}} = \check{\mathrm{G}}_{\mathrm{X},as}/\check{\mathrm{G}}_{\mathrm{X}}$$

  where $\check{\mathrm{G}}_{\mathrm{X}}$ is the dual group of X and $\check{\mathrm{G}}_{\mathrm{X},as} \subset \check{\mathrm{G}}$.
- We'll assume for simplicity $\check{\mathrm{G}}_{\mathrm{X},as} = \check{\mathrm{G}}$.

Dual symmetric variety of X

- The centralizer of $x \in \mathrm{X}^{rss}(F)$ sits in an exact sequence

$$1 \longrightarrow \mathrm{H}_x \longrightarrow T \longrightarrow A_{\mathrm{X}} \longrightarrow 1.$$

- We have the dual SES

$$1 \longrightarrow \check{A}_{\mathrm{X}} \longrightarrow \check{T} \longrightarrow \check{\mathrm{H}}_x \longrightarrow 1.$$

and a diagram

$$\begin{array}{ccccc} \check{A}_{\mathrm{X}} & \longrightarrow & \check{T} & \longrightarrow & \check{\mathrm{H}}_x \\ \downarrow & & \downarrow & & \downarrow \\ \check{\mathrm{G}}_{\mathrm{X}} & \xrightarrow{\varphi_{\mathrm{X}}} & \check{\mathrm{G}} & \longrightarrow & \check{\mathrm{X}}. \end{array}$$

- $\check{T}$ acts on $\check{\mathrm{X}}$ through $\check{\mathrm{H}}_x$
- **Tate-Nakayama duality:** $\kappa \in H^1(F, \mathrm{H}_x)^D \cong \pi_0(\check{\mathrm{H}}_x^{\Gamma})$
- ranging over possible embeddings, obtain $\check{\mathrm{G}}_{\mathrm{X}}$-orbit $[\kappa] \subset \check{\mathrm{X}}$.

Descendents at Semi-simple points

- We thus obtain semi-simple elements $\kappa \in \check{\mathrm{X}}$ and a descent diagram

$$\begin{array}{ccccc} \check{\mathrm{G}}_{\mathrm{X}_\kappa} & \longrightarrow & \check{\mathrm{G}}_\kappa & \longrightarrow & \check{\mathrm{X}}_\kappa \\ \downarrow & & \downarrow & & \downarrow \\ \check{\mathrm{G}}_{\mathrm{X}} & \longrightarrow & \check{\mathrm{G}} & \longrightarrow & \check{\mathrm{X}}. \end{array}$$

Theorem (L)

Suppose that $x \in \mathrm{X}^{rss}(F)$ and $\kappa \in H^1(F, \mathrm{H}_x)^D$. There exists a F-rational symmetric variety $\mathrm{X}_\kappa = \mathrm{G}_\kappa / \mathrm{H}_\kappa$ (unique up to isomorphism) of the quasi-split endoscopic group G_κ such that the top row is dual to X_κ.

- Gives us a notion of endoscopic space
- Existence over $\overline{F}$ is straightforward. Rationality of X_κ is not obvious.

Point comparison and Pre-stabilization

Proposition (L)

Suppose that $\mathrm{X} = \mathrm{G}/\mathrm{H}$ and $\mathrm{X}_\kappa = \mathrm{G}_\kappa/\mathrm{H}_\kappa$ as before. There is a matching of stable semi-simple orbits

$$\phi_\kappa : (\mathrm{X}_\kappa /\!/ \mathrm{H}_\kappa)(F) \longrightarrow (\mathrm{X} /\!/ \mathrm{H})(F).$$

- This allows us to compare stable orbits of X with those on endoscopic variety X_κ.
- (In preparation) Can effect the pre-stabilization

$$TF_{\mathrm{X}}^{ell}(f) = STF_{\mathrm{X}}^{ell}(f) + \sum_{\mathcal{E}} \prod_{v} \mathrm{O}_x^\kappa(f_v)$$

  with sum over **elliptic relative endoscopic data.**

Needed for stabilization: a general theory of transfer factors and local harmonic analytic results

Transfer factors and smooth transfer

Now let's return to $\mathrm{X}_n = U_{2n}/U_n \times U_n$ and $\mathrm{X}_{a,b} = (U_{2a}/U_a \times U_a) \times (U_{2b}/U_b \times U_b)$.

Proposition (L)

1. There is a good notion of **transfer factor** Δ on the regular ss locus.
2. (smooth transfer) For "regularly supported" test functions $f \in C_c^\infty(\mathrm{X}_n(F))$, there exists $f^{a,b} \in C_c^\infty(X_{a,b}(F))$ such that

$$\mathrm{SO}_{(x_a,x_b)}(f^{a,b}) = \Delta(x,(x_a,x_b))\mathrm{O}_x^\kappa(f).$$

- Requires some pure inner forms the varieties $\mathrm{X}_a \times \mathrm{X}_b$ and X_n.
- Regular support constraint comes from the definition of Δ: uses morphism

$$\mathrm{X}_n \longrightarrow \mathcal{H}erm_n$$

The fundamental lemma

Theorem (L)

For any $x \in \mathrm{X}_n^{rs}(F)$ and $\kappa \in H^1(F, \mathrm{H}_x)^D$, there is a decomposition $n = a + b$ such that

$$\mathrm{SO}_{(x_a,x_b)}(\mathbf{1}_{\mathrm{X}_a(\mathcal{O}_F)} \otimes \mathbf{1}_{\mathrm{X}_b(\mathcal{O}_F)}) = \Delta(x,(x_a,x_b))\mathrm{O}_x^{\kappa}(\mathbf{1}_{\mathrm{X}_n(\mathcal{O}_F)}).$$

Follows from reduction to the "Lie algebra" $T_e\mathrm{X}_n(F) \cong \mathrm{Mat}_{n\times n}(E)$

Theorem (L)

For any $x \in \mathrm{Mat}_{n\times n}(E)^{rss}$ and character κ, there exists $a + b = n$ such that

$$\mathrm{SO}_{(x_a,x_b)}(\mathbf{1}_{\mathrm{Mat}_{a\times a}(\mathcal{O}_E)} \otimes \mathbf{1}_{\mathrm{Mat}_{b\times b}(\mathcal{O}_E)}) = \Delta(x,(x_a,x_b))\mathrm{O}_x^{\kappa}(\mathbf{1}_{\mathrm{Mat}_{n\times n}(\mathcal{O}_E)}).$$

Proof sketch

We prove this by reducing to the fundamental lemma for the full Hecke algebra associate to the stabilization of the trace formula

$$\mathrm{Y}_n = \mathrm{GL}_{n,E}/U_n$$

with respect to the **endoscopic symmetric varieties**

$$\mathrm{Y}_{a,b} = \mathrm{GL}_{a,E}/U_a \times \mathrm{GL}_{b,E}/U_b.$$

- Series of reductions of orbital integrals via limiting process and uncertainty principle.
- ultimately reduce to a new comparison of RTFs.

First Reduction

Using the map

$$\mathrm{GL}_{n,E} \xrightarrow{r} \mathrm{Y}_n \subset \mathcal{H}erm_n,$$
$$X \longmapsto X\overline{X}^T,$$

we can analyze

$$\mathrm{O}^{\kappa}_{X}(\mathbf{1}_{\mathrm{Mat}_{n\times n}(\mathcal{O}_E)})$$

via orbital integrals of the non-compactly supported $r_!(\mathbf{1}_{\mathrm{Mat}_{n\times n}(\mathcal{O}_E)})$.

- has additional $\mathrm{GL}_n(\mathcal{O}_E)$-invariance
- Study κ-orbital integrals for functions in $C_c^\infty(\mathrm{Y}_n)^{\mathrm{GL}_n(\mathcal{O}_E)}$.

Endoscopy for Hermitian symmetric space

We introduce a weighted parabolic descent of spherical Hecke algebra which sits in a diagram

$$\begin{array}{ccc}
\mathcal{H}(\mathrm{GL}_n(E)) & \xrightarrow{\xi_{a,b}} & \mathcal{H}(\mathrm{GL}_a(E)) \otimes \mathcal{H}(\mathrm{GL}_b(E)) \\
\Big\downarrow r_! & & \Big\downarrow r_{a,b,!} \\
C_c^\infty(\mathrm{Y}_n)^{\mathrm{GL}_n(\mathcal{O}_E)} & & C_c^\infty(\mathrm{Y}_a \times \mathrm{Y}_b)^{\mathrm{GL}_a(\mathcal{O}_E)\times \mathrm{GL}_b(\mathcal{O}_E)}.
\end{array}$$

Theorem (L)

For each $\phi \in \mathcal{H}(\mathrm{GL}(V_n))$ and for every matching $(x_a, x_b) \in \mathrm{Y}_a \times \mathrm{Y}_b$ and $x \in \mathrm{Y}_n$

$$\mathrm{SO}_{(x_a,x_b)}(r_{a,b,!}(\xi_{a,b}(\phi))) = \Delta(x,(x_a,x_b))\mathrm{O}_x^\kappa(r_!(\phi)).$$

where Δ is the Langlands–Shelstad transfer factor on $\mathcal{H}erm_n \cong \mathrm{Lie}(\mathrm{U}_n)$. This implies the previous theorem.

Two results on orbital integrals

Result 1: (J. Xiao '19): Endoscopic transfer for $\mathcal{H}erm_n$ occurs as a limit of Jacquet-Rallis transfer

$$\mathcal{H}erm_n \times V_n \longleftrightarrow \mathfrak{gl}_n(F) \times F^n \times (F^n)^*.$$

We show how the FL on Y_n follows from novel explicit transfer statements in Jacquet-Rallis transfer.

Result 2: Using a result of Beuzart-Plessis, we use the Weil representation of $\mathrm{SL}_2(F)$ on $C_c^\infty(F^n \times (F^n)^*)$ to reduce to a comparison of orbital integrals

$$\mathcal{H}erm_n \longleftrightarrow \mathfrak{gl}_n(F)$$

Final reduction and globalization

$$\mathcal{H}(\mathrm{GL}_n(E))$$

$$C_c^\infty(\Upsilon_n)^{\mathrm{GL}_n(\mathcal{O}_E)} \xleftarrow{r_!} \mathcal{H}(\mathrm{GL}_n(E)) \xrightarrow{BC} \mathcal{H}(\mathrm{GL}_n(F)).$$

Theorem (L)

Consider the Jacquet–Rallis transfer between the spaces

$$C_c^\infty(\mathcal{H}erm_n) \text{ and } C_c^\infty(\mathfrak{g}l_n(F)).$$

For any $\phi \in \mathcal{H}(\mathrm{GL}_n(E))$, the functions $r_!(\phi)$ and $BC(\phi)$ are transfers of each other.

Prove this fundamental lemma via a new comparison of RTFs, relying on the results of Feigon–Lapid–Offen on unitary periods.

THANK YOU!!