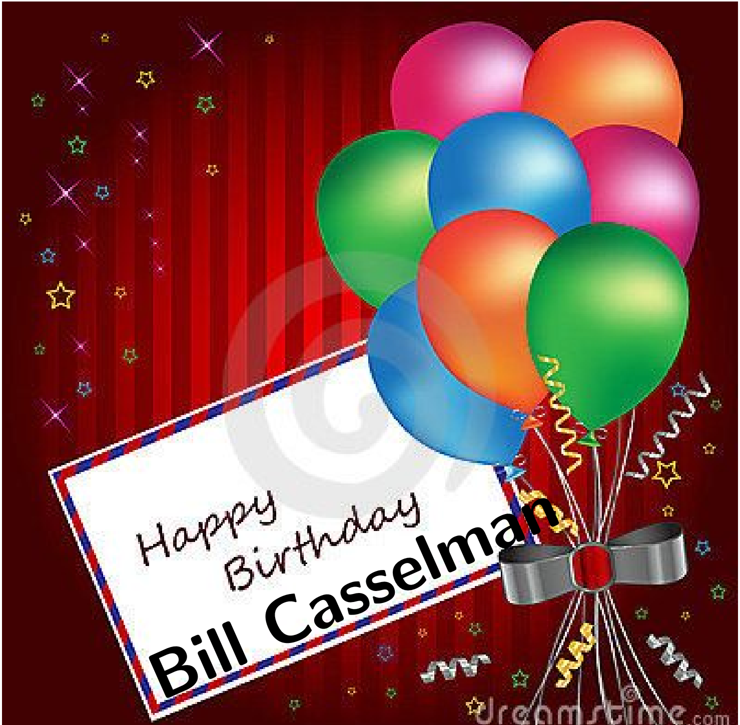

Happy Birthday
Bill Casselman
dreamstime.com

Mathematics Colloquium

Friday, Nov 4 1973

Speaker: Bill Casselman, University of British Columbia

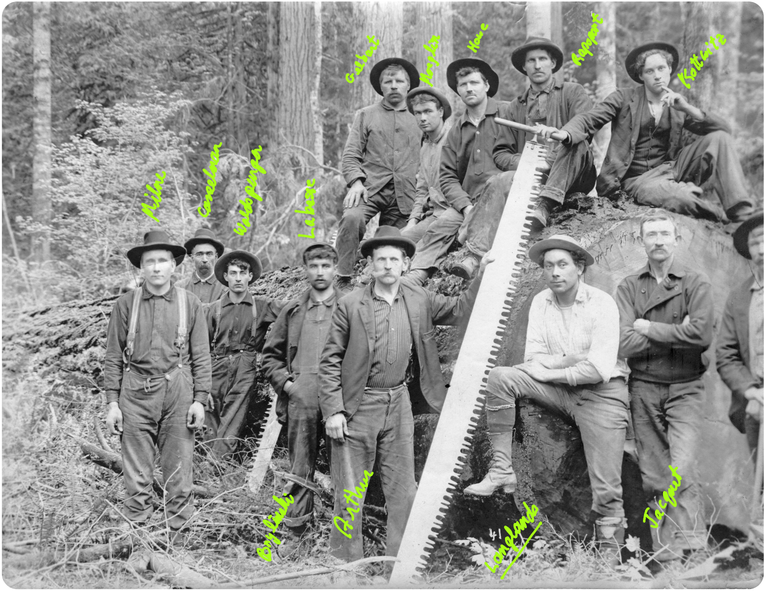
Title: Deligne's Theory of Differential Equations

Location: Barus & Holly 157

Time: 4:30 pm

Coffee and cookies at 4pm in Howell House

We are in a forest whose trees will not fall with a few timid hatchet blows. We have to take up the double-bitted axe and the cross-cut saw, and hope that our muscles are equal to them.



Ordinary points mod p of hyperbolic 3-manifolds

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$G = \mathrm{GSp_{2n}}$

$$X = G(\mathbb{Q})\backslash G(\mathbb{A})/K_f K_\infty$$

Complex points of a Shimura variety that parametrizes principally polarized abelian varieties with level structure.

$d < 0$ square free, $E = \mathbb{Q}[\sqrt{d}]$ quadratic imaginary,
$H = \mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL_2}$

$$Y = H(\mathbb{Q})\backslash H(\mathbb{A})/K_f^H K_\infty^H \sim \coprod_j \Gamma_j\backslash\mathcal{H}_3$$

$\Gamma_j \sim \mathrm{SL_2}(\mathcal{O}_\mathrm{d})$, 3 dimensional hyperbolic manifold, Given $d < 0$ there exists an involution τ_d on $G = \mathrm{GSp_{2n}}$:

τ_d acts on		fixed points
$\mathrm{Sp}_4(\mathbb{R})$		$\mathrm{SL}_2(\mathbb{C})$
$\mathrm{Sp}_4(\mathbb{Q})$		$\mathrm{SL}_2(\mathrm{E})$
$\mathrm{Sp}_4(\mathbb{Z})$		$\mathrm{SL}_2(\mathcal{O}_\mathrm{d})$
$\mathbf{H}_2$		$\mathcal{H}_3$
$G(\mathbb{Q})\backslash G(\mathbb{A})/K$		$\coprod_i H(\mathbb{Q})\backslash H(\mathbb{A}^E)/K^H$

Which abelian varieties lie over X^{τ_d}?

Proposition

The space $X^{\tau_d} = \coprod_i Y_i$ is a coarse moduli space for principally polarized abelian surfaces (A, ω) with level structure and anti-holomorphic multiplication by $\mathcal{O}_d$.

This means:
$\sqrt{d}$ acts anti-holomorphically on A and $\omega(\sqrt{d}x, \sqrt{d}y) = d\omega(x, y)$

A similar story for real structures

There exists an involution τ_0 on $G = \mathrm{GSp_{2n}}$:

τ_0 acts on		fixed points
$\mathrm{Sp_{2n}}(\mathbb{R})$		$\mathrm{GL_n}(\mathbb{R})$
$\mathrm{Sp_{2n}}(\mathbb{Q})$		$\mathrm{GL_n}(\mathbb{Q})$
$\mathrm{Sp_{2n}}(\mathbb{Z})$		$\mathrm{GL_n}(\mathbb{Z})$
$\mathbf{H}_n$		$\mathcal{P}_n$
$G(\mathbb{Q})\backslash G(\mathbb{A})/K$		$\coprod_i \mathrm{GL_n}(\mathbb{Q})\backslash \mathrm{GL_n}(\mathbb{A})/\mathrm{K_f K_\infty}$

Which abelian varieties lie over X^{τ_0}?

Proposition

The space $X^{\tau_0} = \coprod_i Z_i$ is a coarse moduli space for principally polarized abelian varieties (A, ω) with anti-holomorphic involution (that is, real abelian varieties).

$$A = \mathbb{C}^n/L \xrightarrow{\tau} \mathbb{C}^n/L \quad \text{complex anti-linear}$$

Reduction mod p

$$X = G(\mathbb{Q})\backslash G(\mathbb{A})/K_f K_\infty$$

has good reduction $\overline{X}$ at various primes,
which parametrizes principally polarized abelian varieties over $\mathbb{F}_q$.

Kottwitz: sum with $\alpha(\gamma_0;\gamma,\delta) = 1$,

$$\sum_{\gamma_0 \in G(\mathbb{Q})} \sum_{\gamma \in G(\mathbb{A}_f^p)} \sum_{\delta \in G(W_p)} \mathrm{vol}(**)\mathrm{c}(\gamma_0;\gamma,\delta)\mathrm{O}_\gamma(\mathrm{f}^\mathrm{p})\mathrm{TO}_\delta(\phi_\mathrm{p})$$

What happens to the subset $Y = X^{\tau_d}$ when we reduce mod p?

Does $\overline{Y}$ parametrize abelian varieties over $\mathbb{F}_q$ with anti-holomorphic multiplication?

What is anti-holomorphic?

Suppose A is simple, has complex multiplication, say, by $\mathcal{O} \subset L$ and good reduction $\overline{A}$ over $\mathbb{F}_q$.
The Frobenius $\mathrm{Fr_q}$ has a lift to an element $\pi \in L \subset End_{\mathbb{Q}}(A)$

The lift $\pi \in L$ is a Weil q-number:
$\pi\bar{\pi} = q$ for every embedding of $\mathbb{Q}[\pi] \to \mathbb{C}$.
But $\bar{\pi} = q\pi^{-1}$ is a lift of the Vershiebung on $\overline{A}$.
Therefore, complex conjugation on $\overline{A}$, if it is to make sense, should switch the Frobenius and the Vershiebung.

This appears to be nonsense because every morphism will preserve the Frobenius. So we ask:

Q1: Does there exist a "natural" enlargement of the category of abelian varieties over $\mathbb{F}_q$ in which there are new morphisms, including morphisms that exchange the Frobenius with the Vershiebung?

Q2: If so, does there exist a "moduli scheme" of Abelian varieties over $\mathbb{F}_q$ with complex conjugation? with anti-holomorphic multiplication?

For ordinary abelian varieties there is a good answer to Q1.

Recall: $A/\mathbb{F}_q$ is *ordinary*, dim $= n$, iff $A[p] \cong (\mathbb{Z}/(p))^n$
$\iff$ characteristic polynomial is an *ordinary* Weil q-polynomial, (middle coefficient is not divisible by p.)

Theorem of Deligne: There is an equivalence of categories:

$$\{\text{ordinary abelian varieites}/\mathbb{F}_q, \text{rank } n)\} \to \{\text{Deligne modules}(T, F)\}$$
$$A \mapsto (T_A, F_A)$$

$T_A =$ free abelian group of dimension $2n$
$F_A : T_A \to T_A$ char. poly. is an *ordinary Weil q-polynomial*,
there exists $V_A : T_A \to T_A$ with $F_A V_A = V_A F_A = qI$.

[E. Howe]: A polarization $A \to A^\vee$ of corresponds to a rationally nondegenerate symplectic form $\omega : T_A \times T_A \to \mathbb{Z}$ with $\omega(T_A x, y) = \omega(x, V_A y)$ and $R(x, y) = \omega(x, \iota y)$ is symmetric and positive definite.
($\iota =$ totally positive imaginary element of $\mathbb{Q}[F_A]$.)

A morphism $(T,F) \to (T'F')$ of Deligne modules take F to F' but we may consider more general morphisms $T \to T'$.

Definition

Let us say that a *real structure* on a polarized Deligne module (T,F,ω) is an involution $\tau : T \to T$ so that

$$\tau F \tau^{-1} = V, \qquad \omega(\tau x, \tau y) = -\omega(x,y)$$

and anti-holomorphic multiplication is $\mathcal{O}_d \to End(T)$ such that

$$\sqrt{d} \circ F = V \circ \sqrt{d}, \qquad \omega(\sqrt{d}x, \sqrt{d}y) = d\omega(x,y).$$

Proposition

A real structure τ on (T,F,ω) induces involutions τ_ℓ on the Tate modules and an involution τ_p on the Dieudonné module (that switch F and V).

Theorem

There are finitely many isomorphism classes of:
rank $2n$ principally polarized Deligne modules (T, F, ω, τ) with real structure, and principal level N structure ($N \geq 3$).
The number is given by a Kottwitz-like formula.
replacing $\mathrm{Sp_{2n}}$ with $\mathrm{GL_n}$.

There are finitely many isomomrphism classes of principally polarized Deligne modules of rank 4, with level N structure and anti-holomorphic multiplication by $\mathcal{O}_d$.
The number is given by a Kottwitz-like formula replacing $\mathrm{GSp_{2n}}$ with $\mathrm{Res}_{E/\mathbb{Q}} GL_2$.

Isogeny classes (Honda-Tate)

$\mathbb{Q}$ isogeny classes of abelian varities $/\mathbb{F}_q$
$\leftrightarrow \overline{\mathbb{Q}}$ isogeny classes of polarized abelian varieties $/\mathbb{F}_q$
$\leftrightarrow \overline{\mathbb{Q}}$-conjugacy classes $\gamma_0 \in \mathrm{GSp}_{2n}(\mathbb{Q})$,
semisimple, real elliptic, whose characteristic polynomial is a Weil q-polynomial. (First sum in K. formula)

$\overline{\mathbb{Q}}$ isogeny classes of "real" polarized Deligne modules
$\leftrightarrow \mathbb{Q}$-conjugacy classes $A \in \mathrm{GL}_n(\mathbb{Q})$
real elliptic semisimple elements whose characteristic polynomial is ordinary and totally real:

$$b(x) = x^n + \cdots + b_1 x + b_0 = \prod (x - \beta_i) \in \mathbb{Z}[x]$$

- *ordinary* ($\Leftrightarrow p \nmid b_0$)
- *totally real* ($\Leftrightarrow \beta_i$ is totally real)
- $|\beta_i| < \sqrt{q}$.

(In GL_n, $\mathbb{Q}$-conjugacy $= \overline{\mathbb{Q}}$-conjugacy)

The rest of the formula is also interesting
In the case of anti-holomorphic multiplication, there is a restriction on the characteristic polynomial of Frobenius.
Is this total nonsense?
Or do these constructions extend to all abelian varieties over $\mathbb{F}_q$?
Presumably a real structure on $A/\mathbb{F}_q$ is a collection

$$\{\tau_\ell, \tau_p\}$$

of involutions of the Tate and Dieudonné modules which exchange Frobenius and Vershiebung with some compatibility condition, perhaps a Kottwitz-like invariant vanishes?

The End?