Some properties of automorphic forms and a proof of meromorphic continuation of Eisenstein series.

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In my talk I describe the ideas behind the proof of analytic continuation of Eisenstein series by myself and E. Lapid (see arxiv 1911.02342).

The proof is based on some general properties of automorphic forms that are of independent interest.

I will discuss these properties and some extensions of these properties.

One of my goals is to explain that analytic continuation of Eisenstein series is "easy".

By this I mean that it does not require the spectral theory – neither the spectral theory of the automorphic space $\Gamma\backslash G$ nor the spectral theory of self-joint operators. All it requires from Functional analysis is some version of Fredholm theory. In particular, we can work with Banach spaces instead of Hilbert ones.

On the other hand, the knowledge of this analytic continuation is very helpful in description of the spectral decomposition for the automorphic space.

Another goal is to formulate some general properties of automorphic forms that are used in the proof. I think they will have many applications.

We will see that these properties are simply slightly stronger versions of the properties formulated and used in the original Langlands' paper.

1. Set up.

G – locally compact group, usually unimodular, $\Gamma \subset G$ – a discrete subgroup.

We consider the automorphic space $X = \Gamma\backslash G$.. This space has a natural action of the group G, so we can study the corresponding representation $(\Pi, G, F(X))$ of the group G, where $F(X)$ is some space of functions on X (this will be called the automorphic representation).

Fix a global field k and denote by $\mathbf{A} = \mathbf{A}_k$ its Adele ring. Fix a reductive algebraic group $\mathcal{G}$ defined over the field k and consider the automorphic pair (G, Γ), where

$G = \mathcal{G}(A)$ and $\Gamma = \mathcal{G}(k)$.

Let $X = X_G$ denote the automorphic space $X_G = \Gamma\backslash G$. We are interested in the study of the representation $(\Pi, G, F(X))$ where $F(X)$ denotes the space of functions of moderate growth on X

We fix a good maximal compact subgroup $K \subset G$. Then we can decompose our space $F(X)$ with respect to the action of K.

So we will fix some irreducible representation σ of K and study the space $F(X)_\sigma \subset F(X)$

This space is not G-invariant, but we can reconstruct representation of G using Hecke algebra. Namely, consider the Hecke algebra $H(G)$ of smooth compactly supported measures on G. It acts on the space $F(X)$.

Now consider the subalgebra $H_\sigma \subset H(G)$ that preserves K-type σ. It will act on the space $F(X)_\sigma$.

Our main object of study is the subspace of automorphic functions $A(X)_\sigma \subset F(X)$. Many equivalent definitions of automorphic functions.

(*) f is automorphic if the space $H_\sigma(f)$ is finite dimensional

In fact for properties we are interested in we can assume that σ is the trivial representation.

So for simplicity you can assume this to be the case.

2. Parabolic subgroups.

We fix a minimal parabolic subgroup $P_0 \subset G$ and its Levi decomposition $P = M_0 \cdot U_0$. We assume that the subgroups K and M_0 are in a good position.

There is a finite number of subgroups P that contain P_0. They are called standard parabolic.

Every standard parabolic subgroup P has the standard Levi decomposition $P = MU$, where $M \supset M_0$. We call this Levi subgroup $M = M_P$ the standard Levi subgroup.

We denote by Z_P the center of the group M_P.

2.1. **Root system.** Let $L = Hom(M_0, G_m)$ be the lattice of characters of the group M_0. We consider the real vector space $\mathfrak{a} = \mathbb{R} \otimes L$ – the Cartan space of the group G.

In a standard way the dual space $\mathfrak{a}^*$ contains the root system $\Sigma = \Sigma_G$ of the group G with respect to M_0.

All standard parabolic subgroups and the Weyl group $W = Norm(M_0/M_0)$ are described in terms of this root system.

2.2. **Geometric description of the space $\mathfrak{a}^*$.** For any reductive group H over k let us denote $\mathfrak{a}^*(H)$ the group of continuous morphisms $\lambda : H(\mathbf{A}) \to \mathbb{R}^*_+$ trivial on the subgroup $H(k)$. This is a vector space over $\mathbb{R}$.

It is easy to see that $\mathfrak{a}^*(M_0)$ coincides with the dual Cartan space $\mathfrak{a}^*$ described above.

If P is a standard parabolic subgroup, then it is easy to see that $\mathfrak{a}^*(M_P) = \mathfrak{a}^*(Z_P)$ is in a standard way a subspace (end even a direct summand) of the space $\mathfrak{a}^*(M_0) = \mathfrak{a}^*$. We denote this subspace by $\mathfrak{a}^*_P$.

3. Constant term operators.

Let P be a standard parabolic subgroup. We denote by X_P the corresponding automorphic space

$X_P = \Gamma_P \cdot U_P \backslash G$

We consider the space $F(X_P)$ of functions of moderate growth and decompose it into components $F(X_P)_\sigma$. As before we define the subspace $A(X_P)_\sigma$ of automorphic functions on X_P.

We have the **constant term operator**

$C = C_G^P : F(X) \to F(X_P)$ defined by integration over the group $\Gamma_U \backslash U$ via the left action

A function $f \in F(X)$ is called **cuspidal** if $C(f) = 0$. The subspace of cuspidal functions we denote by $F(X)_c$.

We denote by $A(X)_c$ the space of cuspidal automorphic functions.

One of important results by Langlands is that cuspidal automorphic forms rapidly decrease (modulo center of G). This implies that there is the canonical "orthogonal" projection $cusp : A(X) \to A(X)_c$.

4. Cuspidal characters, cuspidal components and cuspidal exponents.

Fix a standard parabolic P. In a similar way we can define the space $A(X_P)_c$ of cuspidal automorphic functions on the space X_P and the cuspidal projection $cusp : A(X_P) \to A(X_P)_c$.

We have the canonical action of the center Z_P on the space X_P on the left. Hence it acts on the space $A(X_P)_c$.

In fact. it is natural to modify this action by a character ρ_P of the group M. We will mostly work with this modified action.

Given an automorphic function f on X_P we consider its cuspidal part $cusp(f)$ and decompose it with respect to characters χ of $Z = Z_P$.

The characters that appear in this decomposition we call cuspidal characters of f ; corresponding automorphic forms f_χ we call cuspidal components of f.

Given a cuspidal character χ we consider its absolute value as a positive valued character of Z, i.e a point e of the vector space $\mathfrak{a}^*(Z) = \mathfrak{a}^*(M) = \mathfrak{a}^*_P \subset \mathfrak{a}^*$. We call such a point a cuspidal exponent of f.

These constrictions allow us to define important invariants for an automorphic form f on X.

Namely, for every standard parabolic subgroup P we can consider its constant term as an automorphic form on X_P and then define cuspidal characters, components and exponents for this form.

4.1. **Leading exponents.** One of important results by Langlands was that any non-zero automorphic form on X has some cuspidal exponents $e(f) \in \mathfrak{a}^*$.

We claim that a stronger statement is correct.

Proposition.

Let f be a nonzero automorphic form on X. Then it has an exponent $e(f) \in \mathfrak{a}^*$ such that $e(f) + \rho$ lies in the Weyl chamber.

Exponents of this type we will call **leading exponents**.

5. Eisenstein series.

The constant term operator

$C = C_G^P : F(X) \to F(X_P)$ has a formal adjoint operator $E = E_P^G : F(X_P) \to F(X)$.

It is given by the summation over $\Gamma_P \backslash \Gamma$ via the left action. We call it the **Eisenstein operator**.

Since this is an infinite sum it does not always absolutely converge. When it converges it commutes with the action of G.

Given an automorphic form ϕ on the space X_P we would like to try to define the automorphic form $f = E(\phi)$ on X.

The standard regularization procedure is as follows.

We include our form ϕ into a family of forms $\phi(s)$ holomorphically depending on some parameter $s \in S$, where S is a connected complex manifold.

We check that for some area of parameters s the function $f(s) = E(\phi(s))$ is well defined and holomorphic in s.

We try to show that the family $f(s)$ has meromorphic extension to all the manifold S.

5.1. **Continuation Principle.** We propose a slightly different regularization procedure.

Namely we would like to characterize the forms $f(s)$ by some properties – some system of equations Ξ_s that holomorphically depends on a parameter s.

Then we show that this system

(i) Has a solution $f(s)$ in some area $S_e \subset S$

(ii) In some area S_u it has no more than one solution.

Then we invoke a very general

Continuation Principle. Under conditions (i) and (ii) for almost every parameter s the system Ξ_s has a unique solution $f(s)$.

These solutions extend to a meromorphic function $s \mapsto f(s)$ on the whole manifold S.

This is a principle, not a theorem. However, I believe that it holds in most natural situations. Of course we have to develop tools to show that it does hold in our case.

6. The system of equations that we use.

Our variety S is a complexification of the space $\mathfrak{a}_P^*$. Every point $s \in S$ defines a character $\chi_s : P \to \mathbb{C}^*$.

We extend this to a right K-invariant function χ_s on the space X_P.

Now we consider the family of automorphic forms $\phi(s) = \chi_s \cdot \phi$ on the space X_P and are trying to characterize the function $f(s) = E(\phi(s))$.

For every s we write the system of equations Ξ_s as follows

(i) $f(s)$ is an automorphic form satisfying the same equations that are satisfied by the form $\phi(s)$.

(ii) All cuspidal characters of $f(s)$ lie on W-orbits of cuspidal characters of $\phi(s)$

(iii) For every cuspidal characters w_χ corresponding to the unit $w = e$ the cuspidal component $f(s)_\chi$ equals to the cuspidal component $\phi(s)_\chi$ of the automorphic form $\phi(s)$.

7. Explanation of the proofs.

Now we have to check 3 different statements.

1. When $Re(s)$ is very dominant the system Ξ_s has a solution defined by absolutely convergent Eisenstein operator E.

2. When $Re(s)$ is very dominant the system Ξ_s has nor more than one solution.

3. The Continuation Principle holds for this system.

Explanation. 1. When $Re(s)$ is very dominant the Eisenstein operator is defined by an absolutely convergent series.

The direct computation using Bruhat decomposition proves item 1. In fact, this is very similar to the proof of Geometric Lemma in local theory.

2. When $Re(s)$ is very dominant all leading exponents will correspond to $w = e$.

Hence the item 2. follows from the fact that an automorphic form is completely described by its leading cuspidal exponents.

8. How to check the continuation principle.

Suppose we have a system of equations of the following type.

We have a complex topological vector space V , and a family W_κ of topological vector spaces W_κ.

Suppose for every index κ we are are given a holomorphic family of morphisms $\nu_\kappa(s) : V \to W_\kappa$ and a holomorphic family of vectors $w_\kappa(s) \in W_\kappa$.

Then we can define a system of equations Ξ_s for a vector $v = v(s) \in V$ by requiring that for every index κ we have

$\nu_\kappa(s)(v) = w(s)$.

We say that the system is of finite type if there exists a holomorphic family of subspaces $L(s) \subset V$ that contains all the solutions of the system Ξ_s.

We say that the system is of locally finite type if for every point s it is of a finite type in some neighborhood of s.

Proposition. Suppose we have a system Ξ_s of locally finite type .Then it satisfies the Continuation Principle.

8.1. **Why our system is of locally finite type.** Langlands has shown that if we fix a K-type and an infinitesimal character then the corresponding space is finite dimensional.

In fact he constructed a finite-dimensional subspace L that contains all the automorphic forms of this type.

By following these arguments more carefully we show that the subspace L can be chosen to holomorphically depend on the parameters (locally).

This means that the system defining automorphic forms is of locally finite type.