On Braverman-Kazhdan/Ngo Program (1)

Casselman's Birthday Conference

Banff (BIRS) - 11/16, 2021

F. Shahidi

Godement - Jacquet

$F = p$-adic $\quad G = GL_n, \quad M_n,$

$C_c^\infty(M_n(F))$ = locally constant with compact support

π = irr. adm. rep. of $GL_n(F)$

$f(x)$ = matrix coefficient of π

$= \langle \pi(x)v, \tilde{v} \rangle \quad v \in \mathcal{H}(\pi), \tilde{v} \in \mathcal{H}(\tilde{\pi})$

$\Phi \in C_c^\infty(M_n(F))$. $\quad \psi \in \hat{F}, \psi \neq 1$.

Fourier transform: $\hat{\Phi}(x) := \int_{M_n(F)} \Phi(y)\psi(tr(xy))\, dy.$

$\check{f}(g) = f(g^{-1}), \ g \in GL_n(F).$

$$Z(\Phi, f, s) = \int_{GL_n(F)} \Phi(x) f(x) |\det x|^s \, dx.$$

$\exists$ a rational fnct. $\gamma^{std}(\pi, s)$ s.t.

$$Z(\hat{\Phi}, \check{f}, (1-s)+\tfrac{n-1}{2}) = \gamma^{std}(\pi,s)\, Z(\Phi, f, s+\tfrac{n-1}{2})$$

$\forall f, \forall \Phi$. (2)

This is completely equivalent to convolving the kernel of the Fourier transform

$$\Phi_\psi(g) = \psi(tr(g))|\det g|^n dg$$

with f :

$$\Phi_\psi * (f\,|\det|^{s+\frac{n-1}{2}}) = \gamma^{std}(\pi,s)\, f\,|\det|^{s+\frac{n-1}{2}},$$

where γ is defined by Schur's lemma.

Braverman – Kazhdan Program is a vast generalization of this to arbitrary reductive G and a finite dimensional representation ρ of ${}^LG$. It has been refined by Ngo, and is being pursued by a number of mathematicians, among them Sakellaridis, Getz, ... Jiang and their collaborators. We refer to L. Lafforgue for his own approach to this problem.

We now discuss what are the replacements for different ingredients: M_n, $\hat{\Phi}$, $S(M_n)$ ..

Replacing M_n

k = alg. closed (Renner, Vinberg)

M = Monoid, affine alg. variety with an associative multiplication and $1 \in M$. We also want M to be normal ($k[M]$ is integrally closed in $k(M)$) which we can achieve by a normalization.

$G = G(M) = M^* =$ units of M.

M is reductive iff $G(M)$ is reductive.

<u>Goal</u>: G = reductive, connected / k - split.

$$\rho : \hat{G} = {}^L G \longrightarrow GL(V_\rho)$$

finite dimensional rep. Want to attach a Monoid M to ρ for which $G = G(M)$.

$\mathcal{W}(\rho)$ = weights of ρ $\quad T \subset G$, maximal torus

$$\rho | \hat{T} = \bigoplus_{\lambda \in \mathcal{W}(\rho)} \lambda .$$

$$\Lambda = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) = X^*(\hat{T})$$

$$\mathcal{W}(\rho) \subset \mathrm{Hom}(\hat{T}, \mathbb{C}^*) = X_*(T) \subset \Lambda_{\mathbb{R}},$$

where $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

$\xi(\rho) = \mathrm{cone}(\mathcal{W}(\rho))$ in $\Lambda_{\mathbb{R}}$; a semi-group

$$:= \left\{ \sum_{\lambda \in \mathcal{W}(\rho)} c_\lambda \lambda \;\middle|\; c_\lambda \geq 0 \right\}.$$

Define $\Lambda^*_{\mathbb{R}} = \Lambda^* \otimes_{\mathbb{Z}} \mathbb{R}$, $\Lambda^* = X^*(T)$

$\Lambda_{\mathbb{R}}$ and $\Lambda^*_{\mathbb{R}}$ are in duality.

$\xi(\rho)^\vee := \text{dual cone} \subset \Lambda^*_{\mathbb{R}}$

$$\sigma^\vee := \xi(\rho)^\vee \cap X^*(T)$$
$$= \text{"rational" dual cone}$$

$k[\sigma^\vee]$ = gp algebra of semi-group $\sigma^\vee$

$$\sigma^\vee \subset k[\sigma^\vee]$$
$$u \longmapsto \chi_u$$

We need a character $\nu : G \longrightarrow \mathbb{G}_m$ s.t.

$$\mathbb{C}^* \xrightarrow{\nu^\vee} \hat{G} \xrightarrow{\rho} GL(V_\rho)$$
$$z \longmapsto z \cdot \mathrm{Id}_\rho$$

This implies $\langle \nu, \omega \rangle = 1$ $\forall \omega \in \mathrm{TNT}(\rho)$.

Thus $\nu \in \sigma^\vee \implies \xi(\rho)$ is <u>strictly</u> convex, i.e., has no lines in it ($\mathrm{cone}(-e_1, e_1) \subset \mathbb{R}^2$ is not, but $\mathrm{cone}(e_1, e_2)$ is - half lines are o.K.). Then $\xi(\rho)$ determines, uniquely, a normal toric variety M_T, a normal affine torus embedding $j : T \subset M_T$. Then

$$M_T = \mathrm{Spec}(k[\sigma^\vee])$$

$X(M_T) = \sigma^\vee$, character semigroup, generates $k[M]$.

Image of push-forward

$$j^*: X(M_T) \hookrightarrow X(T)$$

consist of dominant characters of T that extend to semigroup morphisms $M_T \to \mathbb{A}^1$.

Moreover, $\nu \in j^*(X(M_T))$.

$W = W(G,T)$ acts on $T, M_T, X(T), X(M_T)$

$\lambda \in X(T)$, dominant & integral, defines a rational rep of G.

<u>Renner's construction of M</u>

Choose $\{\lambda_i\}_{i=1}^s$ so that $\bigcup_{i=1}^s W.\lambda_i$ generates $X(M_T)$. Let $(\mu_{\lambda_i}, V_{\lambda_i})$ be rep. attached to λ_i. Set

$$\mu = \bigoplus_{i=1}^s \mu_i \quad , \quad V = \bigoplus_{i=1}^s V_{\lambda_i}$$

We may take $\mu_1 = \nu$. Define

$$M_1 = \overline{\mu(G)} \subset \mathrm{End}(V).$$

Let M be a normalization of M_1.

The construction behaves well w.r.t. parabolic induction.

Example Symmetric powers of $GL(2)$ ⑥

$$\rho = \mathrm{Sym}^n : GL_2(\mathbb{C}) \longrightarrow GL_{n+1}(\mathbb{C})$$

$$\mathrm{Wt}(\rho) = \{\mu_i,\ 0 \le i \le n\},\ \mu_i(\mathrm{diag}(x,y)) = x^i y^{n-i}$$

$$\xi(\rho) \cap X_*(T) = \{(m,\ell) \mid m, \ell \geqslant 0 \text{ in } \mathbb{Z},\ m+\ell \in n\mathbb{Z}\}$$

$$\text{dual cone} = \{(a,b) \in \tfrac{1}{n}\mathbb{Z} \times \tfrac{1}{n}\mathbb{Z} \mid a,b \geqslant 0\ \ a-b \in \mathbb{Z}\}$$

$$\sigma^\vee = \mathbb{Z}_{\geqslant 0}\text{-span of } \{(1,0), (0,1), (\tfrac{1}{n}, \tfrac{1}{n})\}$$

If we let x, y, z denote their images in $k[\sigma^\vee]$, then

$$k[\sigma^\vee] = k(x,y,z) = k[X,Y,Z]/(XY - Z^n)$$

and

$$M_T = \mathrm{Spec}\, k(x,y,z) \subset k^3$$

$$T = M_T \cap (k^*)^3$$

$$= \{(t_1, t_2^n t_1^{-1}, t_2) \mid t_i \in k^*, i = 1,2\}$$

<u>Monoid for Sym^n:</u> $(0,1)$ and $(1,0)$ are Wt-conj. and therefore we can take $\lambda_i \in \{(1,0), (\frac{1}{n}, \frac{1}{n})\}$ as dominant weights:

$$(1,0) \longleftrightarrow \text{std}$$

$$(\tfrac{1}{n}, \tfrac{1}{n}) \longleftrightarrow \nu = \det^{\frac{1}{n}}$$

$$\mu : G \longrightarrow \mathrm{End}(V_{std} \oplus V_{\nu}) = M_2 \times \mathbb{A}^1$$

$$g \longmapsto (g, (\det g)^{\frac{1}{n}})$$

$$M = \overline{\mu(G)} = \overline{\{(g,a) \mid \det g = a^n\}}$$

$$= \mathrm{Spec}\, k[x_1, \dots, x_5]/(x_1 x_4 - x_2 x_3 = x_5^n)$$

$$M^* = GL_2 \times_{\mathbb{G}_m} \mathbb{G}_m = \{(g,a) \mid \det g = a^n\}$$

$$= \begin{cases} GL_2 & n = \text{odd} \\ SL_2 \times GL_1 & n = \text{even} \end{cases}$$

$$\begin{tikzcd}
GL_2 \times_{\mathbb{G}_m} \mathbb{G}_m \arrow[r, "\mathrm{Proj}_1"] \arrow[d, "\nu = \mathrm{Proj}_2"'] & GL_2 \arrow[d, "\det"] \\
\mathbb{G}_m \arrow[r] & \mathbb{G}_m
\end{tikzcd}
\qquad
\begin{tikzcd}
(g,a) \arrow[r, mapsto] \arrow[d, mapsto] & g \arrow[d, mapsto] \\
a \arrow[r, mapsto] & \det g = a^n
\end{tikzcd}$$

$$z \xrightarrow{\nu^\vee} \mathrm{diag}(z^{\frac{1}{n}}, \dots, z^{\frac{1}{n}}) \xrightarrow{\mathrm{Sym}^n} z \cdot I_{n+1}$$

Parabolic induction and Renner's construction:

$P = LN \subset G$, $L \supset T$, $\rho : \hat{G} \longrightarrow GL(V_\rho)$, $\rho_L = \rho|\hat{L}$

$\rho_L \rightsquigarrow M^{\rho_L}$ by Renner. We also have $L \subset G$ and thus $\overline{\mu(L)}$. Then $M^{\rho_L} = \overline{\mu(L)}$

Schwartz functions and Fourier Transforms

One expects a space of ρ-Schwartz functions $S^\rho(G)$ to exist for which GJ can be developed.

Then
$$C_c^\infty(G) \subset S^\rho(G) \subset C^\infty(G),$$
$G = G(k)$. Similarly for a parabolic $P = LN$, $L \supset T$, we should have
$$C_c^\infty(L) \subset S^{\rho_L}(L) \subset C^\infty(L).$$
$S^\rho(G)$ can be connected to M^ρ to be smooth fncs who are supported in the intersection of a compact subset of $M^\rho(k)$ and $G(k)$.

To set up the zeta function we need:

η_G = half the sum of (positive) roots in a Borel sbgp $B \supset T$

λ = h.w. of ρ

$\delta_{G,\rho} = |\nu|^{\langle 2\eta_G, \lambda\rangle}$ $\quad \nu \in X^*(G)$ defined earlier

π = irr. adm. rep of $G(k)$

we absorb $|\nu|^s$ in π and replace $\pi \otimes |\nu|^s$ with π.

f = matrix coefficient of π, $\check{f}(x) = f(x^{-1})$

Given $\Phi \in \mathcal{S}^{\rho}(G)$, set

$$Z(\Phi, f) = \int_{G(k)} \Phi(g) f(g)\, \delta_{G,\rho}(g)\, dg.$$

To set up the other side of the functional equation we need a Fourier transform

$$\mathcal{J}^{\rho} : \mathcal{S}^{\rho}(G) \longrightarrow \mathcal{S}^{\rho}(G).$$

Then we set

$$\tilde{Z}(\mathcal{J}^{\rho}\Phi, \check{f}) = \int_{G(k)} \mathcal{J}^{\rho}\Phi(g)\, \check{f}(g)\, \delta_{G,\rho}(g)\, |\nu(g)|\, dg.$$

Note that $\check{f}$ is a matrix coefficient of $\tilde{\pi}$ and $|\nu|$ takes account for $s \longmapsto 1-s$. Then

$$\tilde{Z}(\mathcal{J}^{\rho}\Phi, \check{f}) = \gamma(\pi, \rho) Z(\Phi, f),$$

for a scalar $\gamma(\pi, \rho)$.

<u>Basic function</u> (Sakellaridis terminology).

Basic fnc Φ_0 attached to an unramified π_0 must satisfy:

$$Z(\Phi_0, f_0) = L(\pi_0, \rho), \qquad (k \in G(\mathcal{O}_k))$$

where $f_0(g) = \langle \pi(g)v, \tilde{v} \rangle$, $\langle v, \tilde{v} \rangle = 1$, $\pi_0(k)v = v$, $\tilde{\pi}_0(k)\tilde{v} = \tilde{v}$. Here $L(\pi_0, \rho)$ is Langlands' "unramified" L-function. It is in fact the inverse Satake transform of $L(\pi_0, \rho)$.

Therefore it is clear that we need $\mathcal{S}^\rho(G)$ and J^ρ on it. The only cases we know these over local fields are GJ and PS-Rallis's doubling method (W.Li, JLZ) some quadratic spaces due to Getz and his Collaborators. I will try to comment what can be accomplished under some basic assumptions.

To proceed assume we know

$$J^\rho : C_c^\infty(G) \longrightarrow C^\infty(G)$$

and define a "provincial" Schwartz space.

$$\mathcal{S}^\rho(G) := C_c^\infty(G) + J^\rho(C_c^\infty(G)),$$

$G = G(k)$. One nice property satisfied by functions in $\mathcal{S}^\rho(G)$ is:

Uniform Smoothness. Let K be an open compact subgroup of G and take $\Phi \in \mathcal{S}^\rho(G)$. Then the space spanned by $\{ k_1 \Phi k_2 \mid \forall k_1, k_2 \in K \}$, where

$(k_1 \Phi k_2)(x) := \Phi(k_2 x k_1)$, <u>is finite dimensional.</u>

Another fact that can be proved using Satake isomorphisms and J^ρ for tori, defined by Ngo, and a GL(1) calculation, is

<u>Proposition</u> 1 (Sh - Sokurski) <u>The basic function</u> Φ_0 <u>belongs to</u> $S^\rho(G)$, <u>our provincial Schwartz space.</u>

This is interesting since the full Schwartz space, as conjectured by Braverman-Kazhdan, is supposed to be larger than this.
I will sketch the proof of this later.
Next we will discuss:
<u>ρ-HC = ρ-Harish-Chandra transform</u>

$P = LN$, Parabolic, TCL.
HC/Satake transform:

$$\Phi_P(l) = \delta_P^{-1/2}(l) \int_{N(k)} \Phi(nl)\, dn$$

We define δ_{L,ρ_L} as we define $\delta_{G,\rho}$ by

$$\delta_{L,\rho_L} = |\nu_L|^{\langle 2\eta_L, \lambda_1 + \cdots \lambda_r \rangle}$$

λ_i highest weights of ρ_L. and set

$$\nu_{G/L} = \nu_{G/L,\rho} := \delta_{G,\rho} / \delta_{L,\rho_L}.$$

Then ρ-HC transfer is

$$\Phi_P^\rho(l) := \nu_{G/L,\rho}^{1/2}(l)\, \Phi_P(l).$$

We expect the following diagram is commutative for all G and ρ:

$$\begin{tikzcd} C_c^\infty(G) \arrow[r, "J^\rho"] \arrow[d, "\rho-HC"'] \arrow[dr, phantom, "\checkmark"] & J^\rho(C_c^\infty(G)) \arrow[d, "\rho-HC"] \\ C_c^\infty(L) \arrow[r, "J^{\rho_L}"] & J^{\rho_L}(C_c^\infty(L)) \end{tikzcd} \quad (*)$$

Then $(*)$ extends to $\Phi_P^\rho : \mathcal{S}^\rho(G) \longrightarrow \mathcal{S}^{\rho_L}(L)$, and validity of $(*)$ implies multiplicativity as we now explain:

σ = irr. adm. rep. of $L(k)$.

Let:

$$\Pi = \mathrm{Ind}_{P(k)}^{G(k)} \sigma \otimes 1$$

Let

$\gamma(\Pi, \rho) = \gamma$ - factor attached to Π_0 and ρ, Π_0 = any irr. constituent of Π.

Then we have:

<u>Multiplicativity</u>: <u>Assume</u> (*) <u>is commutative</u>. Then

$$\gamma(\sigma, \rho_L) = \gamma(\Pi, \rho).$$

Multiplicativity is very important in any theory of L-functions and this gives a general proof of it in this important context.

<u>Sketch of a proof of Proposition 1</u>

1) $G = GL_1$ $\quad M = M_1$ $\quad J^\rho$ = standard F.T. on k.

ψ = unramified $\quad J^\rho \Phi(y) = \int_k \Phi(x)\psi(xy)\,dx$

$\int_{O_k} dx = 1 \quad \Phi_0 = \text{b.f.} = \mathrm{Char}(O_k)$

$\Phi = \mathrm{char}(O_k^*) \quad P_k = $ max. ideal of O_k

$$\boxed{\Phi_0 = \frac{1}{q-1}\,\mathrm{char}(P_k^{-1} \setminus O_k) + \frac{q}{q-1}\hat{\Phi}},$$

i.e., $\Phi_0 \in \mathcal{S}^{std}(GL_1)$. $q = |\mathcal{O}_k/\mathfrak{p}_k|$

2) $G = T_n = (GL_1)^n$ = max torus of GL_n.

Then $\boxed{\Phi^{std}_{T_n} = f_1 + J^{std}(f_2)}$ $f_i \in C_c^\infty(T_n(k))$

follows from 1). Moreover f_i, $i=1,2$, are $W(GL_n, T_n) = W_n$ - invariant.

3) $G = GL_n$. By Satake iso: $\Phi^{std}_{T_n}$

$$\mathcal{H}(GL_n(k), GL_n(\mathcal{O}_k)) \overset{Sat}{\simeq} \mathcal{H}(T_n(k), T_n(\mathcal{O}_k))^{W_n}$$

$$\boxed{\Phi^{std}_{GL_n} = Sat^{-1}(f_1) + Sat^{-1}(J^{std}(f_2))}$$

(*) is proved by GJ. Thus

$$\boxed{\begin{aligned}\Phi^{std}_{GL_n} &= Sat^{-1}(f_1) + J^{std}(Sat^{-1}(f_2)) \\ &\in \mathcal{S}^{std}(GL_n(k))\end{aligned}}$$

4) G = general reductive gp T = maximal torus ρ = rep of $\hat{G}$ μ_i = weights of ρ $\rho_T = \rho|\hat{T}$ $n = \dim\rho$. we have $\rho_T = \mu_1 \oplus \cdots \oplus \mu_n$

dualize: $\tilde{\rho}_T : \mathbb{G}_m^n \longrightarrow T$

$$(x_1, \dots, x_n) \longmapsto \mu_1(x_1)\cdots\mu_n(x_n)$$

$U =$ kernel of $\widetilde{\rho}_T$ in $\mathbb{G}_m^n$.

Define $h_\psi : k^n \longrightarrow \mathbb{C}^*$

$$(x_1, \cdots, x_n) \longmapsto \psi(x_1 + \cdots + x_n)$$

<u>Then Ngo defines:</u> $J^{\rho_T}(t) = \int_{U(k)} h_\psi(xt)\, dx$

Set $\rho_* =$ Push-forward of $\widetilde{\rho}_T$

Then

$$\begin{array}{ccc} S^{std}(T_n) & \xrightarrow{\rho_*} & S^{\rho_T}(T) := \rho_*(S^{std}(T_n)) \\ {\scriptstyle J^{std}} \downarrow & \checkmark & \downarrow {\scriptstyle J^{\rho_T}} \\ S^{std}(T_n) & \xrightarrow{\rho_*} & S^{\rho_T}(T) \end{array} \qquad (**)$$

and

$$\rho_*(\Phi^{std}_{T_n}) = \Phi^{\rho}_T$$

$$= \rho_*(f_1) + \rho_*(J^{std}(f_2)) \overset{(**)}{=} \rho_*(f_1) + J^{\rho_T}(\rho_*(f_2))$$

Thus $\boxed{\Phi^{\rho}_T \in C_c^\infty(T(k)) + J^{\rho}T(C_c^\infty(T(k)))}$

Finally, let

$$J_0^\rho = J^\rho \,|\, \mathcal{H}(G(k), G(O_k)).$$

Then

$$\begin{tikzcd}
\mathcal{H}(GL_n(k), GL_n(\mathcal{O}_k)) \arrow[r, "\rho_* \cdot \mathrm{Sat}", "\simeq"'] \arrow[d, "J^{\mathrm{std}}"'] & \mathcal{H}(T(k), T(\mathcal{O}_k))^W \arrow[r, "\mathrm{Sat}^{-1}"] \arrow[d, "J^{\rho_T}"] & \mathcal{H}(G(k), G(\mathcal{O}_k)) \arrow[d, "J_0^{\rho}"] \\
\mathcal{H}(GL_n(k), GL_n(\mathcal{O}_k)) \arrow[r, "\simeq", "\rho_* \cdot \mathrm{Sat}"'] & \mathcal{H}(T(k), T(\mathcal{O}_k))^W \arrow[r, "\mathrm{Sat}^{-1}"'] & \mathcal{H}(G(k), G(\mathcal{O}_k))
\end{tikzcd}$$

Thus

$$\Phi^{\rho} = \mathrm{Sat}^{-1}(\Phi^{\rho_T}) = \mathrm{Sat}^{-1}(\rho_*(f_1)) + \mathrm{Sat}^{-1}(J^{\rho_T}(\rho_*(f_2)))$$

$$= \mathrm{Sat}^{-1}(\rho_*(f_1)) + J_0^{\rho_T}(\mathrm{Sat}^{-1}(\rho_*(f_2)))$$

Therefore

$$\Phi^{\rho} \in C_c^{\infty}(G(k)) + J^{\rho}(C_c^{\infty}(G(k)))$$

or $\quad \Phi^{\rho} \in \mathcal{S}^{\rho}(G).$

W. Casselman, Symmetric Powers and the Satake transform, Bull. Iranian Math. Soc. 43, No. 4, (2017), 17–54.

BC Ngo, Hankel transform..., Jap. J. Math., 2020.

Bouthier, Ngo & Sakellaridis, Amer. J. Math., Igusa Mem. Vol., 2016.

HAPPY BIRTHDAY BILL!