Residue distributions and spherical Eisenstein series

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The basic residue lemma

- Let V be an oriented Euclidean vector space of dimension n, with complexification $V_{\mathbb{C}}$.
- Let $\mathcal{A}$ be a finite arrangement of affine hyperplanes $H \subset V$, with complexification $\mathcal{A}_{\mathbb{C}}$.
- Let $P(V_{\mathbb{C}})$ denote the space of Paley-Wiener functions on $V_{\mathbb{C}}$, that is $\varphi \in P(V_{\mathbb{C}})$ iff φ is entire and and $\exists\ R > 0$, and for every $N \in \mathbb{N}$, $\exists C_N > 0$ such that for all $z \in V_{\mathbb{C}}$ we have $|\varphi(z)| \leq C_N(1 + \|z\|)^{-N} e^{R\|\mathrm{Re}(z)\|}$.
- We denote by $P(V_{\mathbb{C}})^R$ the space of functions φ holomorphic on $\{z \in V_{\mathbb{C}} \mid \mathrm{Re}(z) < R\}$, and such that for every $N \in \mathbb{N}$, $\exists C_N > 0$ such that $|\varphi(z)| \leq C_N(1 + \|z\|)^{-N}$

- Let ω be a rational $(n,0)$-form on $V_{\mathbb{C}}$ whose singular locus and zero locus is contained in $\mathcal{A}_{\mathbb{C}}$.
- Let $b \in V \setminus \cup_{H\in\mathcal{A}} H$ and let $X^{\omega,b} : P(V_{\mathbb{C}}) \to \mathbb{C}$ be the linear functional on $P(V_{\mathbb{C}})$ defined by

$$X^{\omega,b}(\varphi) := \int_{\mathrm{Re}(z)=b} \varphi(z)\omega(z).$$

- Such linear functionals $X^{\omega,b}$ (or slight variations thereof) often arise in harmonic analysis on reductive groups, in the study of “residual contributions” to the spectrum. Our first goal is a basic decomposition theorem for $X^{\omega,b}$ in terms of tempered distributions with certain support conditions.

- For $H \in \mathcal{A}$, let $n_H \in \mathbb{Z}$ denote the order of ω along $H_{\mathbb{C}} = H + iV_H$. For $L \in L(\mathcal{A})$, the intersection semilattice of $\mathcal{A}$, we define
$$n_L = \sum_{H \in \mathcal{A} : L \subset H} n_H.$$
- We call an affine subspace $L \subset V$ ω-residual if
  (1)
  $$L = \bigcap_{H \in \mathcal{A} : L \subset H \text{ and } n_H < 0} H$$
  (intersection of the pole hyperplanes containing L).
  (2) We have $o_L := -n_L - \mathrm{codim}(L) \geq 0$.
- Examples:
  (1) V itself is a residual subspace.
  (2) If $H \in \mathcal{A}$ with $n_H < 0$ then H is residual ($o_H = -n_H - 1 \geq 0$).

- If $L \in L(\mathcal{A})$ is residual, then we define $V_L \subset V$ as the linear subspace underlying the affine subspace $L \subset V$, and $V^L = (V_L)^\perp$ (the subspace spanned by the lines orthogonal to the hyperplanes of poles $H \in \mathcal{A}$ such that $L \subset H$).
- We define $c_L = V^L \cap L$, the center of L (the point in L with the shortest distance to $0 \in V$). Let $\mathcal{C} \subset V$ be the (finite) set of centers of the ω-residual subspaces.
- $L^{temp} := c_L + iV_L \subset c_L + iV \subset V_{\mathbb{C}}$, the tempered form of L.

Proposition[Heckman, O.]

There exists a unique collection of tempered distributions $X_c^b \in \mathcal{S}'(c + iV)$ with $c \in \mathcal{C}$ such that

(a) $\mathrm{Supp}(X_c^b) \subset \bigcup_{L \text{ residual } : c_L = c} L^{temp}$.

(b) For all $\varphi \in P(V_{\mathbb{C}})$ we have: $X^{\omega,b}(\varphi) = \sum_{c \in \mathcal{C}} X_c^b(\varphi|_{c+iV})$.

- Observe that $\varphi|_{c+iV} \in \mathcal{S}(c+iV)$, hence the expression $X_c^b(\varphi|_{c+iV})$ is meaningful.
- Example: Let $V = \mathbb{R}$ and $\omega = \frac{dx}{x-c}$ with $c \in \mathbb{R}$.
  - If $c \neq 0$ then $X_c^b = \mathrm{sign}(c) 2\pi i \delta_c$ if c separates b and 0, and $X_c^b = 0$ otherwise. Moreover $X_0^b = (x-c)^{-1}|_{i\mathbb{R}}$.
  - If $c = 0$ and $\pm b > 0$ then $X_0^b = \mathrm{Pf}(x^{-1}|_{i\mathbb{R}}) \pm \pi i \delta_0$.

A case of interest

- Now let $\hat{G} \supset \hat{B} \supset \hat{T}$ be a connected reductive group over $\mathbb{C}$, with Borel subgroup $\hat{B}$ and maximal torus $\hat{T}$. Let $V \subset \hat{\mathfrak{g}}$ be the real span of the cocharacter lattice of $\hat{T}$. Let $\Sigma^\vee$ be the root system of $\hat{G}$.
- Define a rational function on V by $c(\lambda) = \prod_{\alpha \in \Sigma_+^\vee} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$.
- Consider the following functionals: For $\varphi \in P(V_{\mathbb{C}})$ and b deep in the Weyl chamber, define:

$$X^b(\varphi) = \int_{\lambda \in b+iV} \varphi(\lambda)\omega^X(\lambda) := (2\pi i)^{-n} \int_{\lambda \in b+iV} \varphi(\lambda) \frac{d\lambda}{c(-\lambda)}$$

and

$$Y^b(\varphi) = \int_{\lambda \in b+iV} \varphi(\lambda)\omega^Y(\lambda) := (2\pi i)^{-n} \int_{\lambda \in b+iV} \varphi(\lambda) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

Symmetrization and the distributions X and Y

Observe the following identity of rational functions: $\sum_{w\in W}\frac{1}{c(-w\lambda)}=\frac{|W|}{c(\lambda)c(-\lambda)}$. This identity and geometric considerations (using the ambient space $V_{\mathbb{C}}$!) yield:

Theorem ("hidden" symmetry of the X-distribution)

- Let $f\in P(V_{\mathbb{C}})$. For every $c\in V_+$ and $w\in W$ we have

$$X^b_{wc}(f|_{wc+iV})=Y^b_c((A_{wc}(f)\circ w)|_{c+iV})$$

  where $A_{wc}(f)\in P(V_{\mathbb{C}})$ is defined by (for $\lambda\in V^{reg}_{\mathbb{C}}$): $A_{wc}(f)(\lambda)=\frac{1}{|W_{wc}|}\sum_{u\in W_{wc}}c(u\lambda)f(u\lambda)$ (the symmetrization operator).
- Moreover, X^b is symmetric in the sense that for all $f\in P(V_{\mathbb{C}})$ we have (with $A(f):=A_0(f)$): $X^b(f)=X^b(A(f))$.

Positivity and regularity of Y^b

Y^b is much better behaved than X^b:

Theorem[Simplicity of Y-poles]

For all $L \subset V$, affine subspace, let $o_L^Y = -n_L^Y - \text{codim}(L)$ with n_L^Y the pole order of ω^Y along L. Then $o_L^Y \leq 0$. In particular, L is ω^Y-residual iff $o_L^Y = 0$ (we say: "order 0"), or equivalently:

$$|\{\alpha \in \Sigma \mid \alpha^\vee|_L = 1\}| = |\{\alpha \in \Sigma \mid \alpha^\vee|_L = 0\}| + \text{codim}(L)$$

Theorem[Heckman, O.]

Let $\mathcal{C}^Y \subset V$ denote the set of centers of ω^Y-residual subspaces (a finite W-invariant set). For all $c \in \mathcal{C}^Y$, Y_c^b is a sum over the ω^Y-residual L such that $c_L = c$ of nonnegative smooth measures $d\nu_L'$ supported by L^{temp} (explicitly known).

The support theorem

Algebraic description of the support of the Y_c:

Theorem

For all $c \in \mathcal{C}_+^Y = \mathcal{C}^Y \cap V_+$, there exists $w \in W$ such that $Y_{wc}^b \neq 0$. In this case, the weight $w(c)$ is in the "anti-Casselman" cone, i.e. the dual chamber of V_+.

Support Theorem of Y^b in terms of nilpotent orbits

We have $c \in \mathcal{C}_+^Y$ iff there exists a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^\vee$ such that $c = \lambda_{\mathcal{O}}$, where $\lambda_{\mathcal{O}}$ is half the weighted Dynkin diagram of $\mathcal{O}$. Hence there is a canonical bijection between $W \backslash \mathcal{C}^Y$ and the set of nilpotent orbits of $\mathfrak{g}^\vee$.

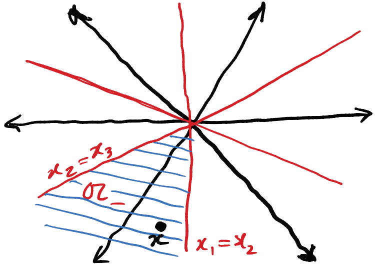
Interpretation: Bose gas with attractive delta potential

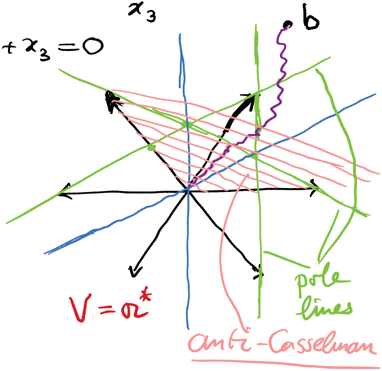
1-dimensional Bose gas with attractive δ-potential

$$\mathcal{H} = -\Delta - \sum_{\alpha \in \Sigma} \delta_0 \alpha$$

Type A_2:

$x_1 + x_2 + x_3 = 0$





Bose gas with attractive delta potential

- The 1-dimensional Bose gas with attractive delta-function potential is completely integrable. Its joint eigenfunctions are $\{E^{YS}(\lambda; x) \mid \lambda \in W \backslash V_{\mathbb{C}}\}$, with for $\lambda \in V_{\mathbb{C}}^{reg}$ and $x \in \mathfrak{a}_{-}$:

$$E^{YS}(\lambda; x) := A_0(e^{(\cdot,x)})(\lambda) = \frac{1}{|W|} \sum_{w \in W} c(w\lambda) e^{w(\lambda,x)} \qquad (1)$$

  and extended W-invariantly to $x \in \mathfrak{a}$. It is W-invariant and holomorphic in λ, of moderate growth in vertical strips.
- Wave packet operator $\theta^{YS} : P^R(V_{\mathbb{C}}) \to L^2(V, dx)^W$ is given by $P^R(V_{\mathbb{C}}) \ni f \to \theta_f^{YS}$ with $\theta_f^{YS}(x) := X^b(f.E^{YS}(\cdot; x))$.

The inner product of wave packets

- We define an anti-linear involution $f \to f^-$ on $P^R(V_{\mathbb{C}})$ by $f^-(\lambda) = \overline{f(\overline{\lambda})}$.
- Given $f \in P^R(V_{\mathbb{C}})$ define (another symmetrization operator):

$$R_f^{YS}(\lambda) := \sum_{w \in W} c(-w\lambda) f^-(-w\lambda)$$

- Inner product ($f, g \in P^R(V_{\mathbb{C}})$, $R > 0$ sufficiently large):

$$\begin{aligned} \langle \theta_f^{YS}, \theta_g^{YS} \rangle &= X^b(g.R_f^{YS}) \\ &= \sum_{L\ \omega^Y-\text{residual}} \int_{L^{temp}} \overline{A(f)(\lambda)} A(g)(\lambda) d\nu_L^{YS}(\lambda) \end{aligned}$$

where the collection $\{d\nu_L^{YS}\}$ consists of smooth positive measures, and is W-equivariant (and explicitly known).

- We will apply our knowledge of these residue distributions to handle residues of unramified spherical Eisenstein series.
- This residual spectrum has of course been studied deeply in the work of Jacquet, Langlands, Moeglin, Waldspurger, Kim, and more recently S. Miller.
- This is joint work in progress with M. De Martino and V. Heiermann (see our preprint arXiv:1512.08566).
- There was unfortunately a gap in arXiv:1512.08566. We think that we have fixed the gap in the proof, but the proof now involves some case by case verifications for the exceptional cases, for which we need to use Maple. Let me describe our current approach and where we are.

Unramified spherical Eisenstein series

Let G be split connected reductive over a number field F. Let $K \subset G(\mathbb{A})$ be maximal compact, and $B = TU$ an F-Borel subgroup. In view of the Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$ we have a left $B(F)$ and right K invariant map $m_B : G(\mathbb{A}) \to T(\mathbb{A})^1 \backslash T(\mathbb{A}) \simeq X_*(T) \otimes \mathbb{R}_+$. Put $\mathfrak{a}^*_{\mathbb{C}} = X^*(T) \otimes \mathbb{C}$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $g \in G(\mathbb{A})$ one defines:

$$\mathcal{E}(\lambda, g) = \sum_{\gamma \in B(F)\backslash G(F)} m_B(\gamma g)^{\lambda+\rho},$$

the Borel Eisenstein series.

Unramified spherical Eisenstein series: Basic facts

Theorem[Langlands]

- Absolutely convergent if $\mathrm{Re}(\lambda - \rho) > 0$, $\in A(G(F)\backslash G(\mathbb{A}))^K$.
- Has meromorphic continuation to $\mathfrak{a}^*_{\mathbb{C}}$ as function of λ.
- Put Λ for the completed Dedekind zeta function of F, and $\rho(s) = s(s-1)\Lambda(s)$ (entire, zeroes in critical strip). For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we put $r(\lambda) = \prod_{\alpha\in\Sigma_+} \rho(\alpha^\vee(\lambda))$ and $c(\lambda) = \prod_{\alpha\in\Sigma_+} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$. Then for all $w \in W$ we have:

$$\mathcal{E}(w\lambda, g) = \frac{c(w\lambda)r(w\lambda)}{c(\lambda)r(\lambda)}\mathcal{E}(\lambda, g)$$

- $\mathcal{E}(\lambda, \cdot)$ is an $\mathcal{H}(G(\mathbb{A})//K)$-eigenform with eigenvalue χ_λ.
- For $f \in P(\mathfrak{a}^*_{\mathbb{C}})^R$ ($R >> 0$), the Pseudo-Eisenstein series $\theta_f := \int_{\mathrm{Re}(\lambda)=b>>0} f(\lambda)\mathcal{E}(\lambda, \cdot)d\lambda \in L^2(G(F)\backslash G(\mathbb{A}))^K$.

Normalized unramified spherical Eisenstein series

Definition

- Define the normalized Eisenstein series by $\mathcal{E}_0(\lambda, g) := \frac{1}{|W|} A_0(r(\cdot)\mathcal{E}(-\cdot, g))(-\lambda) = \frac{1}{|W|} c(-\lambda) r(-\lambda) \mathcal{E}(\lambda, g)$. Then $\mathcal{E}_0$ extends to a holomorphic, W-invariant function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, of moderate growth in vertical strips.
- The normalized pseudo Eisenstein series for $f \in P^R(V_{\mathbb{C}})$: $\theta_f^0(g) := \int_{\mathrm{Re}(\lambda)=b>>0} f(\lambda)\mathcal{E}_0(\lambda, g) \frac{d\lambda}{c(-\lambda)} = X^b(f.\mathcal{E}_0(\cdot; g))$.
- Fix $R >> 0$. We define $L^2(G(F)\backslash G(\mathbb{A}))^K_{[T,1]}$ (or simply $L^{2,K}_{[T,1]}$) as the closure in $L^2(G(F)\backslash G(\mathbb{A}))^K$ of the span of the pseudo-Eisenstein series $\{\theta_f \mid f \in P^R(\mathfrak{a}_{\mathbb{C}}^*)\}$.
- We define $L^{2,K}_{[T,1],0} \subset L^{2,K}_{[T,1]}$ as the closure in $L^2(G(F)\backslash G(\mathbb{A}))^K$ of the span of $\{\theta_f^0 \mid f \in P^R(\mathfrak{a}_{\mathbb{C}}^*)\}$.

Basic challenges

Problem

Give the spectral decomposition of the unitary representation $L^{2,K}_{[T,1]}$ of the abelian $*$-algebra $\mathcal{H}(G(\mathbb{A})//K)$.

We split this in two parts:

Partial problems

- Give the spectral decomposition of the unitary representation $L^{2,K}_{[T,1],0}$ of $\mathcal{H}(G(\mathbb{A})//K)$.
- Show that $L^{2,K}_{[T,1]} = L^{2,K}_{[T,1],0}$.

Residues of unramified Eisenstein series

Theorem[Langlands]

For $f, g \in P^R(\mathfrak{a}^*_{\mathbb{C}})$ $(R >> 0)$ one has the inner product formula

$$(\theta_f, \theta_g) := X^b(gR_f)$$

with $R_f(\lambda) := \sum_{w \in W} c(-w\lambda)\frac{r(\lambda)}{r(w\lambda)} f^-(-w\lambda)$, and $f^-(\lambda) := \overline{f(\overline{\lambda})}$.

Observe: Since R_f is meromorphic in general, it is now not clear that we can express (θ_f, θ_g) in the local distributions $X^b_c(gR_f)$ as in the Yang System case. Rather we are forced to express $X^b_c(gR_f)$ as a sum of integrals of "iterated residues". Similarly, the "hidden symmetry" of X^b is not at all clear. Therefore we first consider the simple situation of the normalized Eisenstein series.

Spectral decomposition of $L^{2,K}_{[T,1],0}$

Theorem[Langlands formula for normalized Eisenstein series]

For $f, g \in P^R(\mathfrak{a}^*_{\mathbb{C}})$ $(R >> 0)$ one has the inner product formula

$$(\theta^0_f, \theta^0_g) := X^b(gR^{YS}_f)$$

with $R^{YS}_f(\lambda) := \sum_{w \in W} c(-w\lambda) f^-(-w\lambda)$ and $f^-(\lambda) := \overline{f(\overline{\lambda})}$ as before. So $\theta^{YS}_f \to \theta^0_f$ defines an isometry $L^2(V, dx)^W \to L^{2,K}_{[T,1],0}$ with the Yang system.

Unramified anti-tempered global Arthur parameters

Let C_F denote the Idèle class group of F. Define:

$$AP^{su}_{[T,1]} := \{\psi : C_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee \;|\; \begin{array}{l}(a)\ \psi|_{C_F} \text{ is bounded.}\\ (b)\ \psi|_{C_F} \text{ factors through } \|\cdot\|.\\ (c)\ \psi|_{\mathrm{SL}_2(\mathbb{C})} \text{ is algebraic.}\end{array}\}$$

Remark

Let $\overline{AP}^{su}_{[T,1]}$ be the set of equivalence classes in $AP^{su}_{[T,1]}$. Given $\psi \in AP^{su}_{[T,1]}$ we can choose $\psi' \in AP^{su}_{[T,1]}$ with $\psi' \sim \psi$ such that:

- For all $\xi \in C_F$, $\psi'(\xi) = \|\xi\|^{\nu'} \in T^\vee$ for a (unique) $\nu' \in i\mathfrak{a}^*$,
- For all $a \in \mathbb{C}^\times$, $\psi'(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) \in T^\vee$.

Arthur parameters and residual spaces

Proposition[De Martino, Heiermann, O.]

Define

$$D : \overline{AP}^{su}_{[T,1]} \to W \backslash \mathfrak{a}^*_{\mathbb{C}}$$
$$\overline{\psi} \to \nu' + d\psi'(\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix})$$

where $\psi' \sim \psi$ and ν' are as above. Then D defines a bijection between $\overline{AP}^{su}_{[T,1]}$ and
$\Xi := W \backslash W\mathrm{Supp}(X^b) = W \backslash \bigcup_{L \text{ residual}} (L^{temp}) \subset W \backslash \mathfrak{a}^*_{\mathbb{C}}$.

Theorem[De Martino, Heiermann, O.]

The Hilbert subspace $L^{2,K}_{[T,1],0} \subset L^{2,K}_{[T,1]}$ is isomorphic to the space $L^2(\Xi, \mu_0)$ for an explicitly known positive measure μ_0 on Ξ, smooth on each component of Ξ.

Corollary[De Martino, Heiermann, O.]

For any distinguished nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}^\vee$, the normalized Eisenstein series $\mathcal{E}_0(\lambda_{\mathcal{O}}, \cdot)$ is a nonzero element in $L^{2,K}_{[T,1],0}$, with explicit L^2-norm.

Corollary[De Martino, Heiermann, O.]

The correponding local representations $\pi_{\nu,\lambda_{\mathcal{O}}}$ of $G(F_\nu)$ are unitarizable at all local places ν of F.

Theorem* (Pending certification for some lines in E_8-spectrum):

$L^{2,K}_{[T,1]} = L^{2,K}_{[T,1],0}$.

Discussion and Approach

- We first rewrite $(\theta_f, \theta_g)_T := X^b(gR_f)_T$ as a sum of integrals over the pole spaces L of X^b (only those!) of iterated residues of the kernel, with their base points arbitrarily close to the centers c_L of L. As in Langlands's analysis, we truncate integrals to $|\mathrm{Im}(\lambda)|^2 \leq T + |\mathrm{Re}(\lambda)|^2$ for some $T >> 0$.
- Next we prove A_{W_c}-symmetry of sum of the contributions at each center c by comparison with $X^b(gR_f^{YS}) = (\theta_f^0, \theta_g^0)$.
- Together this implies the result provided all kernels are holomorphic where we move contours, except for the "algebraic" poles of X^b. (Admissibility, discussed later).

Moeglin's idea to use induction

- Moving the contours for $X^b(gR_f)$ admissibly directly is too hard. Whatever we tried, "computer says no".
- An idea of Moeglin in the classical case: Take an appropriate proper Levi subgroup $G' \subset G$, and assume by induction that the inner product of pseudo Eisenstein series for G' is given by the "Yang System" spectral measure for G', supported on the G' residual pole spaces.
- This gives already partially symmetrized (over W', the Weyl group of G') kernels in X^b, which behave less wild than the kernels of X^b.
- Restricting to G split: We reduce to simple types. The pairs $(G'^\vee, G^\vee)$ we considered are: (X_{n-1}, X_n) for X of classical type, and (D_5, E_6), (E_6, E_7), (E_7, E_8), (C_3, F_4) and (A_1, G_2).

Rewriting $X^b(gR_f)$: The initial integrals

By induction: $(\theta_f, \theta_g)_T = X^b(g.R_f)_T$ as a sum of integrals of the form:

$$(\theta_f, \theta_g)_T =_T \sum_{L' \in \mathcal{L}'_+} \int_{(p_{L,\infty} + iV_L)_{\leq T}} A'_0(g.R_f)\omega^L(\lambda)$$

where ω^L is the residue along L of the W'-symmetrized form ω of ω_X:

$$\omega := \frac{d\lambda}{c'(\lambda)c(-\lambda)}$$

and where $\mathcal{L}'_+$ denotes a set of representatives of the set $\mathcal{L}'$ of residual pole spaces for $G'^\vee$ which are in standard position (so $\mathcal{L}'_+$ is in bijection with the set of nilpotent orbits of $\mathfrak{g}'^\vee$); finally, $p_{L,\infty} = c_L + it\mathfrak{w}' \in L$ with $\mathfrak{w}'$ the unique fundamental coweight orthogonal to $\Sigma'^\vee$, and $t >> 0$.

The initial contour shifts

The factor of the kernel in front of ω^L has a nice form:

$$\begin{aligned} &|W'|A_0'(\psi.R_\phi)(\lambda) \\ &= \left(\sum_{u\in W'} c'(u\lambda)\frac{r(u\lambda)}{r(\lambda)}\psi(u\lambda)\right)\left(\sum_{w\in W} c(-w\lambda)\frac{r(\lambda)}{r(w\lambda)}\phi(-w\lambda)\right) \\ &=: \Sigma'(\psi)\Sigma(\phi) \end{aligned}$$

We first move in each initial integral the base point $p_{L,\infty}$ to a point b_L close to c_L along a generic curve, and then make a symmetrization for the full Weyl group W_{c_L} with A_{c_L} at c_L.

Theorem

$A_0'(\psi.R_\phi)(\lambda)$ holomorphic in a neighbourhood $\sigma_L := [p_{L,\infty}, c_L]$.

The cascade of contour shifts

- Same for residues along ω^L-pole spaces $M \subset L$ of codimension 1 in L such that $\sigma_L \cap M \neq \emptyset$. Put $i_{M,\sigma_L} := \sigma_L \cap M$ (initial point).
- If M is subresidual (i.e. there exists a residual subspace N such that $M^{temp} \subset N^{temp}$) move base point i_{M,σ_M} of residue integral over $i_{M,\sigma_L} + iV_M$ to c_M along $\sigma_M = [i_{M,\sigma_L}, c_M]$.
- If M is not subresidual, move i_{M,σ_L} along $\sigma_M = [i_{M,\sigma_L}, f_M]$ to a (well chosen) $f_M \in M$ such that at a prior stage we had a residue integral over $u(f_M + iV_M)$ for some $u \in W'$.
- This stops in finitely many steps. The cascade C is a collection of pairs (σ, M) with M a ω-pole space and $\sigma \subset M$ a segment, representing the set of W'-orbits of such pairs encountered in such algorithm.

ω-pole spaces L with $o_L^\omega = 0$ ("simple poles")

Let M be an ω-pole space, and $\sigma \subset M$ such that $\exists u \in W'$ such that $u(\sigma, M) \in C$ (we say: M appears in C). For a base point $b \in \sigma$ we have a residue integral of the form

$$\int_{(b+iV_M)_{\leq T}} \mathrm{Res}_M(\Sigma'(\psi)\Sigma(\phi)\omega)$$

in which the kernel is a residue datum of order o_M^ω.
If $o_M^\omega = 0$ then this simplifies to

$$\int_{(b+iV_M)_{\leq T}} ((\Sigma'(\psi)\Sigma(\phi))|_M)\omega^M$$

In general a cascade contains several levels (up to 5 for (E_7, E_8)) (2 for classical cases), and pole space of higher order (up to order 3 for (E_7, E_8)).

Admissible cascades

Definition

A cascade C is called admissible if there exist subsets $\mathrm{Adm}(L) \subset L$ for all $L \in C$ such that:

- $\mathrm{Adm}(L)$ is a nonempty closed convex set.
- $\Sigma(\lambda)\Sigma'(\lambda)|_L$ is holomorphic on $\mathrm{Adm}(L) + iV_L$.
- For all initial pole spaces $L \in \mathcal{L}_+$ we have $p_{L,\infty} \in \mathrm{Adm}(L)$.
- If $L \in C$ is residual then $c_L \in \mathrm{Adm}(L)$.
- If $(\sigma, L) \in C$ and $M \subset L$ is an ω-pole with $\sigma \cap M \neq \emptyset$ then $\mathrm{Adm}(L) \cap M \subset \mathrm{Adm}(M)$.

Moving to the center

Theorem*

There exists an admissible cascade C (pending certification that $c_L \in \mathrm{Adm}(L)$ for a W-orbit of residual lines for E_8) such that we can move, for each pole space $L \in C$, all base points to a single point $b_L \in \mathrm{Adm}(L)$ - which is close to c_L if L is subresidual - without creating new residues.

Such movement of a base point in a segment $\sigma \subset L$ is not guaranteed by $\sigma \subset \mathrm{Adm}(L)$ if $o_L^\omega > 0$. Fortunately, we found a C such that all poles L with $o_L^\omega > 0$ are met in c_L with only 3 exceptions for E_8, two of which are easy to deal with. For the remaining case (one residual line L of type $E_7(a4)$) it turns out that the potential strip of critical poles is disjoint from the strip around L^{temp} containing the spherical complementary series).

Comparison with $X^b(\theta)$ for $\theta \in P^R(V_{\mathbb{C}})$

We can now write the contribution of each pole space in C as a single residue integral. Comparison with the case $X^b(\theta)$ for $\theta \in P^R(V_{\mathbb{C}})$ (using the same contour shifts in C) is quite powerful now:

- The contribution of a non-subresidual pole space L cancels.
- The sum of the contributions at a residual center c have additional symmetry for the operator A_{Wc}.

Theorem*

$$(\theta_\phi, q_T(\theta_\psi)) = \sum_{L \in W \backslash \mathcal{L}} |W| \int_{L^{temp}_{\leq T}} A_0(r(\cdot)\psi)(\lambda)\overline{A_0(r(\cdot)\phi)(\lambda)} \frac{d\nu_L(\lambda)}{r(-\lambda)r(\lambda)}$$

Happy birthday, Bill!