Relations between cusp forms sharing Hecke eigenvalues

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To Bill Casselman with admiration on his 80th anniversary
I hope he continues to come back to India!

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1. INTRODUCTION

Let F be a number field. Each automorphic representation π of $\mathrm{GL}_d(\mathbb{A}_F)$ gives rise to Hecke eigenvalues (also called the Satake parameter), a d-tuple of (unordered) nonzero complex numbers $H(\pi_v) = (a_{1v}, \cdots, a_{dv})$, at each place v of F where π is unramified, and thus at almost all places of F.

Let π_1 and π_2 be two irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ which are written as isobaric sums:

$$\pi_1 = \pi_{11} \boxplus \pi_{12} \boxplus \cdots \boxplus \pi_{1\ell},$$
$$\pi_2 = \pi_{21} \boxplus \pi_{22} \boxplus \cdots \boxplus \pi_{2\ell'},$$

where π_{1j} and π_{2k} are irreducible cuspidal (unitary) automorphic representations of $\mathrm{GL}_{d_j}(\mathbb{A}_F)$ and $\mathrm{GL}_{d_k}(\mathbb{A}_F)$ respectively. Then by the strong multiplicity one theorem due to Jacquet and Shalika, if π_1 and π_2 have the same Hecke eigenvalues $H(\pi_{1,v}) = H(\pi_{2,v})$ at almost all places v of F where π_1, π_2 are unramified, then $\ell = \ell'$, and up to a permutation of indices, $\pi_{1j} = \pi_{2j}$.

In this lecture, we will consider a variant of the strong multiplicity one theorem, identified in the following definition.

Definition: Given automorphic representations π_1 on $\mathrm{GL}(m, \mathbb{A}_F)$ and π_2 on $\mathrm{GL}(n, \mathbb{A}_F)$, we say that π_1 is immersed in π_2, written $\pi_1 \preceq \pi_2$, if the Hecke eigenvalues of π_1 (counted with multiplicity) are contained in the Hecke eigenvalues of π_2 (counted with multiplicity) for almost all primes of the number field F.

On the other hand, we say that π_1 is embedded in π_2, written as $\pi_1 \subset \pi_2$ if there is an automorphic representation π_3 such that,

$$\pi_2 = \pi_1 \boxplus \pi_3.$$

The following is the motivating question for this lecture.

Question:

(1) Can it happen for distinct cuspidal representations that $\pi_1 \preceq \pi_2$?
(2) If yes, can we classify all such pairs of cuspidal representations $\pi_1 \preceq \pi_2$?

One would have liked to assert that for cuspidal representations $\pi_1 \preceq \pi_2$ never happens if $\pi_1 \neq \pi_2$, but that is not true. For example, let π be a cuspidal non-CM automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$. At any unramified place v of F, if (a_v, a_v^{-1}) are the Hecke eigenvalues of π_v, then for the automorphic representation $\mathrm{Sym}^2(\pi)$ of $\mathrm{PGL}_3(\mathbb{A}_F)$, the Hecke eigenvalues at the place v of F, are $(a_v^2, 1, a_v^{-2})$. Thus the Hecke eigenvalues of the trivial representation of $\mathrm{GL}_1(\mathbb{A}_F)$ are contained in the set of Hecke eigenvalues of the cuspidal automorphic representation $\mathrm{Sym}^2(\pi)$ of $\mathrm{PGL}_3(\mathbb{A}_F)$ at each unramified place of π.

The present work is based on the hope that for cuspidal representations π_1, π_2, although π_1 can be immersed in π_2, without π_1 being the same as π_2, this happens rarely, and only for pairs of representations (π_1, π_2) which are related in some well-defined way, and it is this relationship that we seek to discover!

2. TWO SAMPLE RESULTS

We begin by proving the following simple proposition.

Proposition 1. *Let* π_1 *(resp.* π_2*) be an irreducible cuspidal automorphic representation of* $\mathrm{GL}_m(\mathbb{A}_F)$ *(resp.* $\mathrm{GL}_{m+1}(\mathbb{A}_F)$*). Then* $\pi_1 \not\preceq \pi_2$.

Proof. The proof is a simple consequence of the strong multiplicity one theorem of Jacquet-Shalika recalled at the beginning of this lecture. Let ω_1 (resp. ω_2) be the central character of π_1 (resp. π_2); these are Grössencharacters of $\mathrm{GL}_1(\mathbb{A}_F)$. It is easy to see that, if $H(\pi_{1,v})$ is contained in $H(\pi_{2,v})$ at almost all places v of F, then,

$$\pi_1 \boxplus (\omega_2/\omega_1) = \pi_2,$$

which is not allowed by the strong multiplicity one theorem. $\square$

Here is another similarly 'negative' result, this time proved with some more effort.

Proposition 2. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$. Then $1 \not\preceq \pi$.*

Proof. We will prove the proposition by contradiction, so assume that $H(\pi_v)$ contains 1 at almost all places of F where π is unramified. Observe that to say that $H(\pi_v)$ contains 1 is equivalent to saying that $\det(1 - H(\pi_v)) = 0$, which translates into the following identity (assuming that $H(\pi_v)$ operates on a 4 dimensional vector space V):

$$1 - V + \Lambda^2(V) - \Lambda^3(V) + \Lambda^4(V) = 0.$$

(One way to think of this identity is in the Grothendieck group of representations of an abstract group G which comes equipped with a 4-dimensional representation V of G such that the action of any $g \in G$ on V has a nonzero fixed vector.)

Thus we get the identity:

$$1 + \Lambda^4(V) + \Lambda^2(V) = V + \Lambda^3(V).$$

Let the central character of π be $\omega : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$. Since we know by a work of Kim, that for π automorphic on $\mathrm{GL}_4(\mathbb{A}_F)$, $\Lambda^2(\pi)$ is automorphic on $\mathrm{GL}_6(\mathbb{A}_F)$, by the strong multiplicity one theorem, we get an identity of the isobaric sum of automorphic representations:

$$1 \boxplus \omega \boxplus \Lambda^2(\pi) = \pi \boxplus \omega \cdot \pi^\vee.$$

Observe that the right hand side of this equality is a sum of two cuspidal representations on $\mathrm{GL}_4(\mathbb{A}_F)$, whereas there are two one dimensional characters of $\mathbb{A}_F^\times / F^\times$ on the left hand side. This is not allowed by the strong multiplicity one theorem, therefore the proof of the proposition is completed. $\square$

3. A QUESTION

The following precise question lies at the basis of this work (which one could consider as arising from the most wishful thinking!).

Question 1. Let π_1 (resp. π_2) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ (resp. $\mathrm{GL}_{m+2}(\mathbb{A}_F)$). Suppose that $\pi_1 \preceq \pi_2$. Then is there an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, and a character $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, such that,

$$\begin{aligned} \pi_1 &= \chi \cdot \omega \otimes \mathrm{Sym}^{m-1}(\pi), \\ \pi_2 &= \chi \otimes \mathrm{Sym}^{m+1}(\pi), \end{aligned}$$

i.e., up to twist by a character, is it true that:

$$(\pi_1, \pi_2) = (\omega \otimes \mathrm{Sym}^{m-1}(\pi), \mathrm{Sym}^{m+1}(\pi)).$$

We will provide an affirmative answer to this question for $m = 1, 2, 3$ in this lecture. On the other hand, we will also provide counter-examples to this question using the strong Artin conjecture for all pairs of integers $(q-1, q+1)$ where $q \geq 5$ is a prime power. The work of Calegari proves strong Artin conjecture for certain cases for $q = 5$, which allows us to construct an unconditional counter examples for the pair $(\mathrm{GL}_4, \mathrm{GL}_6)$ over $\mathbb{Q}$.

In spite of the counter-example(s), I would like to think that the question has an answer essentially in the affirmative, as we discuss later.

Remark 1. Here is a geometric analogue of the questions being discussed in this lecture which was settled in a recent work of Khare and Larsen. Let A and B be abelian varieties over a number field F with A simple. For v any finite place of F where both A and B have good reduction, let A_v, B_v denote their reductions mod v (thus A_v, B_v are abelian varieties over finite fields). Assume that there are isogenies from A_v to B_v (not surjective as we are not assuming $\dim(A) = \dim(B)$) for almost all places v of F where A and B have good reduction. Then the question is if there is an isogeny from A to B? If $\dim(A) = \dim(B)$, this is a consequence of the famous theorem of Faltings.

Remark 2. Most of the lecture deals with cusp forms (π_1, π_2) on $(\mathrm{GL}_m(\mathbb{A}_F), \mathrm{GL}_n(\mathbb{A}_F))$ for the restricted pairs (m, n) with $n = m+2$, as the first non-obvious case beyond $m = n$ and $n = m+1$, such that $\pi_1 \preceq \pi_2$ However, one might begin at the other extreme $(m, n) = (1, n)$, and try to classify cuspforms π_2 on $\mathrm{GL}_n(\mathbb{A}_F)$ such that $1 \preceq \pi_2$. By Theorem 1 below, there is a nice answer for $(m, n) = (1, 3)$, whereas by Proposition 2, there are none in the case $(m, n) = (1, 4)$. It is easy to see that the cuspidal representations π_2 of $\mathrm{GL}_6(\mathbb{A}_F)$ which arise as the basechange of a cuspidal selfdual representation of $\mathrm{PGL}_3(\mathbb{A}_E)$ for E/F quadratic, have Hecke eigenvalue 1 at almost all places of F, and a theorem of Yamana can be used to prove a converse. We have not investigated the situation for general pairs $(1, n)$.

4. Theorems

Theorem 1. *If* π *is a cuspform on* PGL_3 *with* $1_{GL_1} \preceq \pi$*, then*

$$\pi = \mathrm{Sym}^2(f),$$

for some f *on* PGL_2*.*

Theorem 2. *If* π_1 *is a cuspform on* PGL_2 *and* π_2 *on* PGL_4 *with* $\pi_1 \preceq \pi_2$*, then*

$$\pi_2 = \mathrm{Sym}^3(\pi_1).$$

Theorem 3. *If* π_1 *is a cuspform on* PGL_3 *and* π_2 *on* PGL_5 *with* $\pi_1 \preceq \pi_2$*, then for some* π *on* PGL_2*,*

$$\pi_1 = \mathrm{Sym}^2(\pi),$$
$$\pi_2 = \mathrm{Sym}^4(\pi).$$

Remark 3. Thus, the first time we do not know how to handle the question is for the pair $(\mathrm{GL}_4, \mathrm{GL}_6)$. We will actually give a counter-example to the question in this case.

5. A FORMAL IDENTITY

Lemma 1. *If π_1 is a cuspform on GL_n and π_2 on GL_{n+2} with central characters ω_1, ω_2, then if $\pi_1 \preceq \pi_2$, we have:*

$$\pi_2 \boxtimes \pi_1 \boxplus \omega_2/\omega_1 = \Lambda^2(\pi_2) \boxplus \mathrm{Sym}^2(\pi_1),$$

considered as a formal identity of Hecke eigenvalues on the two sides.

Proof. The identity in this lemma is nothing but (writing $V = W + [2]$):

$$(W + [2]) \otimes W + \Lambda^2[2] = \Lambda^2(W + [2]) + \mathrm{Sym}^2[W]$$

which is:

$$W \otimes W + [2] \otimes W + \Lambda^2[2] = \Lambda^2(W) + W \otimes [2] + \Lambda^2[2] + \mathrm{Sym}^2[W].$$

□

6. PROOF OF THE THEOREM FOR $(\mathrm{GL}_2, \mathrm{GL}_4)$

In this case, the identity in the previous Lemma becomes:

$$\pi_2 \boxtimes \pi_1 \boxplus 1 = \Lambda^2(\pi_2) \boxplus \mathrm{Sym}^2(\pi_1).$$

An important observation is that since RHS is known to be automorphic by Gelbart-Jacquet and Kim, it follows from the above identity and the Rankin-Selberg theory that the RHS of the above equality must contain the trivial representation of GL_1 as an isobaric direct summand, and therefore $\pi_2 \boxtimes \pi_1$ is automorphic, and the equality above is one of automorphic representations! (Although automorphy of $\mathrm{GL}(2) \times \mathrm{GL}(4)$ is not known in general!).

We will have to deal with 2 cases depending on π_1 is CM or not. Assume it is not CM, then $\mathrm{Sym}^2(\pi_1)$ is cuspidal, hence the above identity forces

$$1 \subset \Lambda^2(\pi_2),$$

and

$$\mathrm{Sym}^2(\pi_1) \subset \pi_2 \boxtimes \pi_1,$$

which is easily seen to be equivalent to (applying Kim-Shahidi on automorphy of $\mathrm{GL}_2 \times \mathrm{GL}_3$ and using the fact that $\pi_1 \otimes \pi_2$ is known to be automorphic which we noted earlier):

$$\pi_2 \subset \pi_1^\vee \otimes \mathrm{Sym}^2(\pi_1) = \mathrm{Sym}^3(\pi_1) + \pi_1,$$

leaving the only option to be:

$$\pi_2 = \mathrm{Sym}^3(\pi_1).$$

7. ARTIN REPRESENTATIONS

Example 1. The finite group $G = \mathrm{PGL}_2(\mathbb{F}_5)$ gives rise to irreducible representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of dimension 4 corresponding to a cuspidal representation of $\mathrm{PGL}_2(\mathbb{F}_5)$ of dimensions $4 = q-1$, and dimension 6 corresponding to a principal series of dimension $6 = q+1$. It can be seen that the cuspidal representation of $G = \mathrm{PGL}_2(\mathbb{F}_5)$ is immersed in the principal series representation. Thus, assuming strong Artin conjecture which is known by the work of Calegari in this case, we find that there are cusp forms π_1, π_2 on $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}}), \mathrm{GL}_6(\mathbb{A}_{\mathbb{Q}})$ with $\pi_1 \preceq \pi_2$ but which do not arise from symmetric powers of a cuspform on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ since in this case, the 4 and 6 dimensional representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ cannot be written as $\mathrm{sym}^3(\sigma), \mathrm{Sym}^5(\sigma)$. (Some details are omitted here.)

8. GROUP THEORETIC CONSIDERATIONS

The question studied in this lecture can be studied from a purely group theoretic point of view, and seems to be of interest. We are unaware of this group theoretic point of view to have been put to use earlier. We will prove that the group theoretic question has an affirmative answer for groups which are not virtually abelian, i.e., do not contain a subgroup of finite index which is abelian.

For representations V_1 and V_2 of a group G, define a relationship $V_1 \preceq V_2$ (read V_1 is immersed in V_2) if for each element $g \in G$, the set of eigenvalues of the action of g on V_1 (counted with multiplicities) is contained in the set of eigenvalues of g acting on V_2 (counted with multiplicities). Thus if $V_1 \subset V_2$ as representations of G, then $V_1 \preceq V_2$. If $V_1 \preceq V_2$ and $\dim(V_1) = \dim(V_2)$, then of course, $V_1 \cong V_2$ as G-modules as they have the same characters.

If $\dim(V_1) + 1 = \dim(V_2)$, then also $V_1 \preceq V_2$ implies $V_1 \subset V_2$ as G-modules. This is because the representations on the two sides of the equality below have the same character:

$$V_2 = V_1 + \det(V_2)/\det(V_1).$$

However, it is not true in general that if $V_1 \preceq V_2$, then $V_1 \subset V_2$ as G-modules as we see in the following proposition.

Proposition 3. *Let* $G = \mathrm{GL}_2(\mathbb{F}_q)$. *Let* C *be an irreducible cuspidal representation of* $\mathrm{GL}_2(\mathbb{F}_q)$ *of dimension* $(q-1)$*, and* P *an irreducible principal series representation of* $\mathrm{GL}_2(\mathbb{F}_q)$ *of dimension* $(q+1)$. *Assume that the central character of* C *and* P *are the same, which is* $\omega : Z = \mathbb{F}_q^\times \to \mathbb{C}^\times$. *Then* $C \preceq P$*, with*

$$\dim P - \dim C = 2.$$

The aim of this work is to understand the relationship $V_1 \preceq V_2$, in the first non-trivial case of $\dim(V_1) = \dim(V_2) - 2$. Let's formulate the following questions.

Question 1: Let G be a compact, possibly disconnected, Lie group of dimension > 0. Classify triples (G, π_1, π_2) of irreducible representations π_1 and π_2 of G with $\pi_1 \preceq \pi_2$, such that G acts faithfully on $\pi_1 + \pi_2$, and

$$\dim(\pi_2) - \dim(\pi_1) = 2.$$

Here is the harder question dealing with finite groups:

Question 2: Let G be a finite group. Classify triples (G, π_1, π_2) of irreducible representations π_1 and π_2 of G with $\pi_1 \preceq \pi_2$, such that G acts faithfully on $\pi_1 + \pi_2$, and

$$\dim(\pi_2) - \dim(\pi_1) = 2.$$

The answer to these questions will have a direct bearing on the question on automorphic forms which is where the questions arose.

9. QUESTION 1

In this section we answer Question 1 completely as long as the group is not 'vitually abelian', i.e., does not contain a finite index subgroup which is abelian.

In the following proposition, we call a connected reductive group Q of type A_1 if its derived subgroup is $\mathrm{PGL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$.

Proposition 4. *Let G be a connected reductive algebraic group over $\mathbb{C}$. Let π_1 and π_2 be two finite dimensional representations of G with $\pi_1 \preceq \pi_2$ such that*

$$\dim(\pi_2) - \dim(\pi_1) = 2.$$

Then there are the following two options:

(1) $\pi_2 = \pi_1 + \lambda + \mu$, where λ, μ are one dimensional representations of G, or

(2) G has a reductive quotient Q of type A_1 and π_1', π_2' irreducible representations of Q of dimensions d and $d+2$ respectively ($d = 0$ allowed) such that,

$$\begin{aligned} \pi_1 &= \pi + \pi_1', \\ \pi_2 &= \pi + \pi_2', \end{aligned}$$

for a finite dimensional representation π of G. '

The following corollary follows by an application of Clifford theory (applied to the normal subgroup G_0 of G) combined with Proposition 4 applied to G_0, we omit its proof.

Corollary 1. *Let G be an algebraic group over $\mathbb{C}$, with G_0, the connected component of identity, a non-abelian reductive group. Assume G has irreducible finite dimensional representations π_1 and π_2, with $\pi_1 \preceq \pi_2$ (when restricted to G_0) such that*

$$\dim(\pi_2) - \dim(\pi_1) = 2,$$

and the action of G on $\pi_1 + \pi_2$ is faithful. Then, both π_1, π_2 remain irreducible when restricted to G_0, and their restriction to G_0 arises as in the previous proposition.

10. FINITE GROUP CASE

In the case of finite groups, we do not know if there are irreducible representations $\pi_1 \preceq \pi_2$ with $\dim(\pi_2) - \dim(\pi_1) = 2$ besides the examples provided by $\mathrm{GL}_2(\mathbb{F}_q)$ in Proposition 1. Perhaps it is not so common to have simple groups with irreducible representations π_1, π_2 with $\dim(\pi_2) - \dim(\pi_1) = 2$ (even without the condition $\pi_1 \preceq \pi_2$!), so I do not expect many examples beyond $\mathrm{GL}_2(\mathbb{F}_q)$.

One of the difficulties in trying to use the condition $\pi_1 \preceq \pi_2$ is that it is not clear it can be represented in terms of "character theory".

However, if $\pi_1 \preceq \pi_2$, $\dim(\pi_2) - \dim(\pi_1) = 2$, we have:

$$\pi_2 \otimes \pi_1 \oplus \omega_2/\omega_1 = \Lambda^2(\pi_2) \oplus \mathrm{Sym}^2(\pi_1),$$

both sides being representations of G of dimension

$$n(n+2) + 1 = (n+1)^2 = (n+1)(n+2)/2 + n(n+1)/2.$$

Thus, if $\pi_1 \preceq \pi_2$, $\dim(\pi_2) - \dim(\pi_1) = 2$, we have in $\pi_2 \otimes \pi_1 \oplus \omega_2/\omega_1 = \Lambda^2(\pi_2) \oplus \mathrm{Sym}^2(\pi_1)$, a necessary condition, which can be checked via character theory, although I am not sure it is necessary and sufficient for $\pi_1 \preceq \pi_2$, $\dim(\pi_2) - \dim(\pi_1) = 2$ to hold.

At the other extreme of the relationship $\pi_1 \preceq \pi_2$ is when $\pi_1 = 1$. In this case, the question amounts to classifying irreducible representations (π, V) of a finite group G, which have the property that every element $g \in G$ fixes a nonzero vector in V. A nice example of $1 \preceq \pi$ is provided by π, the reflection representation of the alternating group A_n, n even.

The best situation to happen of course is that $\pi_1 \preceq \pi_2$ implies $\pi_1 \subset \pi_2$ not only among irreducible representations but for all representations of a given finite group G. But perhaps this never happens beyond cyclic (or abelian?) groups because $\Lambda^k(\mathbb{C}^n) \preceq \mathrm{Sym}^k(\mathbb{C}^n)$ as representations of $\mathrm{GL}_n(\mathbb{C})$, and therefore $\Lambda^k(V) \preceq \mathrm{Sym}^k(V)$ for any representation V of any group G.

11. Remark on $\dim(\pi_2) - \dim(\pi_1) > 2$

Remark 4. By proposition 4, there are no relations $\pi_1 \preceq \pi_2$ among irreducible representations of a connected simple algebraic group with $\dim(\pi_2) - \dim(\pi_1) = 2$, other than the obvious ones for $G = \mathrm{SL}_2(\mathbb{C})$, and $\mathrm{PGL}_2(\mathbb{C})$:

(1) $\pi_1 = \mathrm{Sym}^{d-1}(\mathbb{C}^2)$.
(2) $\pi_2 = \mathrm{Sym}^{d+1}(\mathbb{C}^2)$.

Without the constraint on $\dim(\pi_2) - \dim(\pi_1) = 2$, there are naturally many other representations, such as the pair of representations $\Lambda^k(\mathbb{C}^n), \mathrm{Sym}^k(\mathbb{C}^n)$ of $\mathrm{GL}_n(\mathbb{C})$. It seems interesting to classify all possible pairs $\pi_1 \preceq \pi_2$ for connected simple algebraic groups.

12. Another well-studied question

If G is any finite group, it is a consequence of character theory that if we have two homomorphisms $\phi_1 : G \to \mathrm{GL}_n(\mathbb{C})$ and $\phi_2 : G \to \mathrm{GL}_n(\mathbb{C})$ for which $\phi_1(g), \phi_2(g)$ are conjugate for all $g \in G$, then in fact ϕ_1 and ϕ_2 are conjugate on all of G by a fixed element of $\mathrm{GL}_n(\mathbb{C})$. This concept may be called, "locally conjugate implies globally conjugate". This is, however, specific to $\mathrm{GL}_n(\mathbb{C})$, and if we have some other group, say $\mathrm{PGL}_n(\mathbb{C})$, it is not true that 'locally conjugate implies globally conjugate", and is an interesting and well-studied question.

Our question could be said to be a variant of this question, but now even for $\mathrm{GL}_n(\mathbb{C})$, there seems to be counter-examples.

13. PROOF OF PROPOSITION 2

Let T be a maximal torus in G, and W its Weyl group. Clearly, $\pi_1 \preceq \pi_2$ if and only if the weights of π_1 for the torus T are contained in the weights of π_2 (with multiplicity). Since weights are W-invariant, if $\pi_1 \preceq \pi_2$ with $\dim(\pi_2) - \dim(\pi_1) = 2$, we see that there is a set of two (not necessarily distinct) weights of T (that of $\pi_2 - \pi_1$) which is W-invariant. By Lemma 2 below, this means that either these are weights of T invariant under W, hence arise from characters $\lambda, \mu : G \to \mathbb{C}^\times$, or the group G has a quotient Q (obtained by dividing G by all normal simple groups except one which is $\mathrm{PGL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$), with a quotient S of T as a maximal torus in Q. Proof of the Proposition is easily completed in either case.

Lemma 2. *Let G be a simple algebraic group with T a maximal torus and W its Weyl group. Then if χ is a non-trivial character of T whose W-orbit has ≤ 2 elements, then G is SL_2, or PGL_2.*

Proof. The proof follows from the observation that the stabilizer of χ in W is the Weyl group of a Levi subgroup M of G and $|W_G/W_M| > 2$ if the rank of G is ≥ 2, and $G \neq M$. $\square$

14. A FINAL COMMENT

Given that one has a rather simple proof of the group theoretic analogue of the question on Automorphic representations as long as the group involved is not finite, one wonders what methods in Automorphic forms might be there to make similar conclusions.

Thus in the automorphic context, when one of the representations π_1, or π_2 is Steinberg at a finite place, or has a regular infinitesimal character at one of the archimedean places of F, our question (asserting π_1 and π_2 being symmetric powers of GL_2) should have an affirmative answer, but for the moment, I do not know how to deal with these. A related question is: how does one decide that a cuspform on $\mathrm{GL}_n(\mathbb{A}_F)$ is a symmetric power?

Thank you!