Top cohomology of congruence subgroups of $SL_n \mathbb{Z}$

joint with Jeremy Miller, Andrew Putman

$$\Gamma_n(p) = \ker\left(SL_n\mathbb{Z} \xrightarrow{\text{mod } p} SL_n \mathbb{Z}/p\mathbb{Z}\right)$$

$p \geq 3$ prime

Borel-Serre duality

$$H^{\binom{n}{2}-i}(\Gamma_n(p);\mathbb{Z}) \cong H_i(\Gamma_n(p); St_n \mathbb{Q})$$

Steinberg module

Tits building

F field, $T_n(F)$ simplicial complex

w/ p-simplices

$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq F^n$ flags of subspaces of F^n

Solomon-Tits (1969) $T_n(F) \simeq \bigvee S^{n-2}$

Steinberg module $GL_n F \circlearrowright St_n F := \tilde{H}_{n-2}(T_n(F))$

Cor $H^i(\Gamma_n(p)) = 0$ for $i > \binom{n}{2}$

Question What is $H^{\binom{n}{2}}(\Gamma_n(p))$?

Lee-Szczarba (1976)

$$H^{\binom{n}{2}}(\Gamma_n(p)) \cong St_n(\mathbb{F}_3) \cong \mathbb{Z}^{3^{\binom{n}{2}}}$$

Idea:

$$T_n(\mathbb{Q}) \longrightarrow T_n(\mathbb{F}_p)$$
$$V \longmapsto (V \cap \mathbb{Z}^n) \bmod p$$

$\Gamma_n(p)$-equiv map w/ $\Gamma_n(p) \circlearrowright T_n(\mathbb{F}_p)$ triv.

$$H^{\binom{n}{2}}(\Gamma_n(p)) \cong H_0(\Gamma_n(p); St_n \mathbb{Q})$$

$$= \tilde{H}_{n-2}(T_n(\mathbb{Q}))_{\Gamma_n(p)} \longrightarrow \tilde{H}_{n-2}(T_n(\mathbb{F}_p)) = St_n \mathbb{F}_p$$

Always surjective BUT not injective if $\mathbb{F}_p$ has more units than ± 1.

Better idea: Take instead $T_n(\mathbb{Q}) \longrightarrow T_n(\mathbb{Q})/\Gamma_n(p)$

"oriented" version of $T_n(\mathbb{F}_p)$:
includes $|\mathbb{F}_p^\times/\pm 1|$ copies of subspaces
$0 \subsetneqq V \subsetneqq \mathbb{F}_p^n$.

Note: • $T_n(\mathbb{Q})/\Gamma_n(3) \cong T_n(\mathbb{F}_3)$
• $T_n(\mathbb{Q})/\Gamma_n(p) \simeq \bigvee S^{n-2}$

Main Thm (Miller – P. – Putman)

$$H^{\binom{n}{2}}(\Gamma_n(p)) \longrightarrow \tilde{H}_{n-2}(T_n(\mathbb{Q})/\Gamma_n(p))$$

is surjective for all n and primes p

is injective if and only if $p = 3, 5$

(or $p=2$ or $n \leq 1$)

Rem: • New and complete calculation of $H^{\binom{n}{2}}(\Gamma_n(5))$.

• $\tilde{H}_{n-2}(T_n(\mathbb{Q})/\Gamma_n(p))$ free abelian of rank t_n is extremely computable

$$t_n = \left(\frac{p-3}{2} + \left(\frac{p-1}{2}\right) p^{n-1}\right) \cdot t_{n-1} + \frac{(p-1)(p-3)}{4} \sum_{k=1}^{n-2} p^k \left| Gr_k(\mathbb{F}_p^{n-1})\right| \cdot t_k \cdot t_{n-k-1}$$

(using discrete Morse theory)

• $t_{15} \approx 2 \cdot 10^{78}$ (1 sec)

t_{200} in 1min

• Proof of Main thm even gives

$$\operatorname{rk} H^{\binom{n}{2}}(\Gamma_n(p)) \geq t_n + \frac{(p+2)(p-3)(p-5)(p-1)}{24} \cdot |Gr_2 \mathbb{F}_p^n| \cdot t_{n-2}$$

• Schwermer

$\underline{n=2}$

Modular curve $Y(\Gamma_2(p)) \simeq B\Gamma_2(p)$

surface of genus $\dfrac{(p+2)(p-3)(p-5)}{24}$

and $\left| T_2(\mathbb{Q}) / \Gamma_2(p) \right|$ punctures

$$\Rightarrow H^1(\Gamma_2(p)) \cong \mathbb{Z}^{\frac{(p+2)(p-3)(p-5)}{12}} \oplus \tilde{H}_0\left(T_2(\mathbb{Q})/\Gamma_2(p)\right)$$

cohomology in the interior

cohomology at infinity

Let $\overline{X}_n$ denote the Borel–Serre bordification of the symmetric space $X_n = SL_n\mathbb{R}/SO(n)$

image = cohomology at infinity

$$\begin{tikzcd}
 & H^{\binom{n}{2}}(\Gamma_n(p)) \arrow[d, phantom, sloped, "\cong"] & & & \\
H^{\binom{n}{2}}(\overline{X}_n/\Gamma_n(p), \partial\overline{X}_n/\Gamma_n(p)) \arrow[r] & H^{\binom{n}{2}}(\overline{X}_n/\Gamma_n(p)) \arrow[r] \arrow[ddr, bend right] & H^{\binom{n}{2}}(\partial\overline{X}_n/\Gamma_n(p)) \arrow[r] \arrow[d, phantom, sloped, "\cong\ \text{P.D.}"] & H^{\binom{n}{2}+1}(\overline{X}_n/\Gamma_n(p), \partial\overline{X}_n/\Gamma_n(p)) \arrow[r] \arrow[d, phantom, sloped, "\cong\ \text{P.D.}"] & 0 \\
 & & H_{n-2}(\partial\overline{X}_n/\Gamma_n(p)) \arrow[r] \arrow[d] & H_{n-2}(\overline{X}_n/\Gamma_n(p)) \arrow[d, phantom, sloped, "\cong"] & \\
 & (n\geq 3) & H_{n-2}(T_n(\mathbb{Q})/\Gamma_n(p)) & H_{n-2}(\Gamma_n(p)) &
\end{tikzcd}$$

image = cohomology in the interior

$$\partial\overline{X}_n \xrightarrow{\simeq} T_n(\mathbb{Q})$$

- $\partial\overline{X}_n/\Gamma_n(p) \longrightarrow T_n(\mathbb{Q})/\Gamma_n(p)$ not a homotopy equivalence
- $H_{n-2}(\Gamma_n(p))$ is not generally zero
- cohomology in the interior is not generally zero (but for $n\geq 4$)

Further Questions:

Expand computation to:

- primes $p \geq 7$
- composite numbers
- codim 1 → Miller's talk
- other rings like $\mathbb{Z}[i]$, $\mathbb{Z}[\frac{1+\sqrt{3}i}{2}]$ → Wilson's talk
- congruence subgroups of $Sp_{2n}\mathbb{Z}$ → Brück's talk

Def $BA_n(\mathbb{Z})$ subcpx of n-th Voronoi cpx Vor_n "generated" by simplices in Vor_2:

vertices = unimodular vector $\mathbb{Z}^n/_{\pm 1}$

= $\cancel{rk}$ 1 - summands $\mathbb{Z}^n$

p-simplex $\{\pm v_0, \dots, \pm v_p\}$ if either

$v_0, \dots, v_p$ partial basis of $\mathbb{Z}^n$

or

$v_0 = v_1 + v_2$

$v_1, \dots, v_p$ partial basis of $\mathbb{Z}^n$

Church - Putman: $BA_n(\mathbb{Z})$ $(n-1)$-ctd

n-dim