ON FARRELL–TATE COHOMOLOGY OF GL_3
OVER RINGS OF QUADRATIC INTEGERS

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ABSTRACT. The goal of the present paper is to push forward the frontiers of computations on mod ℓ Farrell–Tate cohomology for arithmetic groups. We deal with ℓ-rank 1 cases different from PSL_2. The conjugacy classification of cyclic subgroups of order ℓ is reduced to the classification of modules of C_ℓ-group rings over suitable rings of integers which are principal ideal domains, generalizing an old result of Reiner. As an example of the number-theoretic input required for the Farrell–Tate cohomology computations, we discuss in detail the homological torsion in PGL_3 over principal ideal rings of quadratic integers, accompanied by machine computations in the imaginary quadratic case.

CONTENTS

1. INTRODUCTION

In the present paper, we investigate the mod ℓ Farrell–Tate cohomology of some arithmetic groups Γ of ℓ-rank 1. This means that all elementary abelian ℓ-groups in Γ are in fact cyclic. Via Brown's formula for Farrell–Tate cohomology, we only need to know the number of conjugacy classes of cyclic subgroups of Γ of order ℓ and the Farrell–Tate cohomology of their normalizers. It is well-known, cf. e.g. the previous computations of mod ℓ Farrell–Tate cohomology for $\mathrm{SL}_2(\mathcal{O}_{K,S})$ in [14], that the conjugacy classification of cyclic subgroups translates in some way into class group questions which can be handled by classical algebraic number theory.

For the groups $\mathrm{SL}_2(\mathcal{O}_{K,S})$ discussed in [14], the modules over the group rings $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$ corresponding to the cyclic subgroups were indecomposable. This is no longer true for the cyclic subgroups in $\mathrm{PGL}_3(\mathcal{O}_{K,S})$, and the interaction between the indecomposable constituents of the corresponding $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules has to be taken into account. On the matrix level, the situation to keep in mind is that of (3×3)-matrices with integer entries which have an upper triangular block form, one block given by a (2×2)-matrix of order ℓ, a further diagonal entry of 1, and the additional new things that need to be handled are the things that appear in the off-diagonal block.

To deal with this situation, we provide a generalization of a result of Reiner [15], concerning the classification of $\mathbb{Z}[\mathrm{C}_\ell]$-modules. We generalize his result to a partial classification of $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules, provided that $\mathcal{O}_{K,S}$ is a principal ideal domain and the ring $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain, cf. Theorem 4.4 and Proposition 4.7. Since the conjugacy classification for cyclic subgroups translates into the isomorphism classification of such modules over the group rings, this extension of Reiner's result can be used to provide computations of numbers of conjugacy classes in a number of interesting cases. We also extend Reiner's result in a different direction: we describe the automorphism groups of the $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules, and this translates into a computation of the centralizers and normalizers of cyclic subgroups in suitable arithmetic groups like $\mathrm{PGL}_3(\mathcal{O}_{-m})$, cf. Section 5.

m	1	2	7	11	15	19	5
λ	0	0	0	0	0	0	1
μ	2	4	3	4	7	3	8

TABLE 1. Machine results on the parameters λ and μ of Theorem 1.1.

With these results in hand, we can then discuss a couple of example cases and provide formulas for the mod ℓ Farrell–Tate cohomology. The cases we consider are mod 3 Farrell–Tate cohomology for rings of quadratic integers, cf. Theorem 6.6 for the imaginary quadratic case and Proposition 6.8 for the real quadratic case. We also discuss the 5-torsion for $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{5})})$, the 7-torsion for $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})})$ and the ℓ-torsion in $\mathrm{PGL}_\ell(\mathbb{Z})$, cf. Section 6. In the imaginary quadratic case, our result describing the homological 3-torsion is the following, for a proof see page 11.

Theorem 1.1. *Let $A = \mathbb{Z} \times \mathbb{Z}/6$, and let $\mathbb{Z}/2$ act on A by sign inversion to construct $A \rtimes \mathbb{Z}/2$. Then for the integers $m \in \{1, 2, 7, 11, 19, 43, 67, 163\}$, the mod 3 Farrell-Tate cohomology of $\mathrm{PGL}_3(\mathcal{O}_{-m})$ admits an isomorphism*
$$\widehat{\mathrm{H}}^\bullet(\mathrm{PGL}_3(\mathcal{O}_{-m}); \mathbb{F}_3) \cong \widehat{\mathrm{H}}^\bullet(A; \mathbb{F}_3)^{\oplus \lambda} \widetilde{\oplus} \widehat{\mathrm{H}}^\bullet(A \rtimes \mathbb{Z}/2; \mathbb{F}_3)^{\oplus \mu},$$
where λ respectively μ are the numbers of conjugacy classes of order-3-subgroups in $\mathrm{PGL}_3(\mathcal{O}_{-m})$ which do not, respectively which do, have a dihedral overgroup in $\mathrm{PGL}_3(\mathcal{O}_{-m})$.

Here, the cohomology rings $\widehat{\mathrm{H}}^\bullet(A; \mathbb{F}_3)$ and $\widehat{\mathrm{H}}^\bullet(A \rtimes \mathbb{Z}/2; \mathbb{F}_3)$ are described explicitly in Proposition 6.7 and the discussion that follows it. The numbers λ and μ can be computed in terms of the Galois-action on the class group of $\mathbb{Q}(\sqrt{-m}, \zeta_3)$ and the action of $\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-m}, \zeta_3)}$ on $\mathcal{O}_{-m}/(3)$, cf. Section 6.

We accompany our study by machine computations on Voronoi cell complexes with $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})})$-action, constructed with Sebastian Schönnenbeck's software [3, 16]. Using a cell subdivision algorithm that was recently introduced by the authors [6], it has been possible to extract the ℓ-torsion subcomplexes. Then for each conjugacy class of cyclic subgroups of order ℓ, we get a connected component of the reduced ℓ-torsion subcomplex, which determines the Farrell–Tate cohomology of its normalizer and tells us whether it contributes to the quantity λ or μ in Theorem 1.1. We will describe the machine computations in Section 7, the results for small quadratic integer rings $\mathcal{O}_{-m}$ being displayed in Table 1.

For the cases $m = 1, 2, 7, 11, 19$, this coincides precisely with the number-theoretic formulas discussed in Section 6. Note that the cases $m = 15$ and $m = 5$ are cases where the conditions of Theorem 1.1 are not satisfied: in both cases, $\mathcal{O}_{-m}$ fails to be a principal ideal domain, and for $m = 15$ an additional requirement for our analysis, $3 \nmid m$, is violated. In these cases, we cannot yet provide general number-theoretic counts for the conjugacy classes of order-3-subgroups, but see the discussion of the case $\mathcal{O}_{-5}$ in Section A

Structure of the paper: A few preliminary statements are made in Section 2, and the translation between conjugacy classes of cyclic subgroups and modules over the group rings for the cyclic groups is explained in Section 3. We discuss a generalization of Reiner's classification of modules over group rings in Section 4. A description of the centralizers and normalizers of cyclic subgroups in terms of the automorphisms of the corresponding modules over the group rings is given in Section 5. Finally, the example computations are given in Section 6, and we discuss the machine computations in Section 7.

2. PRELIMINARIES

2.1. **Finite subgroups.** In the present paper, we will deal with the Farrell–Tate cohomology of arithmetic groups Γ which are of ℓ-rank 1. This means that the maximal abelian ℓ-subgroup of G has rank 1. Note that if the arithmetic group $\Gamma = \mathrm{PGL}_n(\mathcal{O}_{K,S})$ for $n \geq 2$ contains a non-trivial cyclic group C_ℓ, then $[K(\zeta_\ell) : K] \leq n$. In that case, if the ℓ-rank is 1, we must have $[K(\zeta_\ell) : K] > \frac{n}{2}$ (and that already excludes $\ell = 2$). In the specific case $\Gamma = \mathrm{PGL}_3(\mathcal{O}_{K,S})$ this means that the degree $[K(\zeta_\ell) : K]$ is either 2 or 3. In the case where $K = \mathbb{Q}(\sqrt{-m})$ is an imaginary quadratic field, this restricts us to the primes $\ell = 3, 5, 7$. Actually, the case $\ell = 5$ doesn't appear since the quadratic subfield of $\mathbb{Q}(\zeta_5)$ is $\mathbb{Q}(\sqrt{5})$ which is real, and the case $\ell = 7$ only appears for the quadratic subfield $\mathbb{Q}(\sqrt{-7})$ of $\mathbb{Q}(\zeta_7)$.

For example, if there is a 5-torsion element in $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})})$, then the cyclotomic field $\mathbb{Q}(\zeta_5)$ must embed into the matrix algebra $\mathrm{M}_3(\mathbb{Q}(\sqrt{-m}))$ with ζ_5 mapping to the 5-torsion element. This is only possible if $[\mathbb{Q}(\zeta_5, \sqrt{-m}) : \mathbb{Q}(\sqrt{-m})] \leq 2$, because subfields of the 3×3-matrix algebra can only have degree at most 3. The cyclotomic field $\mathbb{Q}(\zeta_5)$ has a unique quadratic subfield which is $\mathbb{Q}(\sqrt{5})$ and therefore the composite $\mathbb{Q}(\zeta_5, \sqrt{-m})$ has degree 8 whenever m is positive. (This can also be checked by

Pari/GP, a minimal polynomial for the composite is given by $x^8 - x^6 + x^4 - x^2 + 1$.) In particular, there can be no 5-torsion elements in GL_3 over imaginary quadratic number rings.

2.2. Farrell–Tate cohomology and Brown's formula.

To compute the relevant examples of Farrell–Tate cohomology of linear groups, we will use Brown's formula for ℓ-rank 1, cf. [4, Corollary X.7.4]. In this case, Brown's complex of elementary abelian ℓ-subgroups of Γ (also known as the Quillen complex) is in fact a disjoint union of conjugacy classes of cyclic ℓ-subgroups of Γ, and the formula is given by

$$\widehat{\mathrm{H}}^\bullet(\Gamma, \mathbb{F}_\ell) \cong \prod_{[G \leq \Gamma], G \text{ cyclic}} \widetilde{\mathrm{H}}^\bullet(\mathrm{C}_\Gamma(G), \mathbb{F}_\ell)^{\mathrm{N}_\Gamma(G)/\mathrm{C}_\Gamma(G)},$$

where the sum on the right is indexed by the conjugacy classes of finite cyclic subgroups of Γ. To evaluate the formula, we need to determine the conjugacy classes of cyclic subgroups as well as the structure of their normalizers. These questions can be translated into questions about the isomorphism classification of modules over groups rings, and the question of automorphism groups of such modules. For cyclic groups, these questions can be approached using the classical work of Reiner, cf. [15].

3. Conjugacy classification of cyclic subgroups

In this section, we will relate the conjugacy classification of cyclic subgroups C_ℓ in general linear groups over S-integer rings $\mathcal{O}_{K,S}$ to the isomorphism classification of modules over the group rings $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$; this is a rather classical argument, cf. [11]. The isomorphism classification will be done in the next section, generalizing Reiner's article [15] on the isomorphism classification of modules over the integral group ring $\mathbb{Z}[\mathrm{C}_\ell]$.

Proposition 3.1. *There is an injection from the set of conjugacy classes of embeddings $\mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ to the set of isomorphism classes of $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules whose underlying $\mathcal{O}_{K,S}$-module is free of rank n. The only isomorphism class not in the image is the one where the C_ℓ-action is trivial.*

Proof. (i) Assume we have a subgroup $\mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$. In particular, we have a non-trivial action of C_ℓ on $M = \mathcal{O}_{K,S}^{\oplus n}$. We use this action to turn M into an $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-module by letting the element $[g]$ for $g \in \mathrm{C}_\ell$ act via the embedding $\mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$.

(ii) Assume we have two subgroups $\phi, \phi' \colon \mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ which are conjugate. Then any conjugating matrix A gives rise to commutative diagrams

$$\begin{array}{ccc}
\mathcal{O}_{K,S}^{\oplus n} & \xrightarrow{\phi(g)} & \mathcal{O}_{K,S}^{\oplus n} \\
{\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\
\mathcal{O}_{K,S}^{\oplus n} & \xrightarrow[\phi'(g)]{} & \mathcal{O}_{K,S}^{\oplus n}
\end{array}$$

showing that the two $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules associated to ϕ and ϕ' are isomorphic via A.

(iii) Conversely, assume we have an $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-module M whose underlying $\mathcal{O}_{K,S}$-module is free of rank n. We choose an $\mathcal{O}_{K,S}$-basis for M. The representing matrices for the automorphisms $[g]$ for $g \in \mathrm{C}_\ell$ provide an embedding $\mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ since by assumption the action of C_ℓ is non-trivial. Different choices of basis will give rise to subgroups which are conjugate via change-of-basis matrices.

(iv) Assume we have an isomorphism $f \colon M \cong M'$ of $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules as in (iii). Then a choice of basis for M induces a choice of basis for M' via f. With these choices of bases, the modules M and M' give rise to the same subgroup of $\mathrm{GL}_n(\mathcal{O}_{K,S})$. The independence-of-basis statement in (iii) implies that the subgroups associated to M and M' (for arbitrary choices of bases) are conjugate. $\square$

Remark 3.2. *It should be pointed out that there is a difference between conjugacy classes of embeddings $\mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ and conjugacy classes of cyclic subgroups of $\mathrm{GL}_n(\mathcal{O}_{K,S})$ of order ℓ. For a non-trivial automorphism $\phi \colon \mathrm{C}_\ell \to \mathrm{C}_\ell$ and some embedding $\iota \colon \mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$, the two embeddings ι and $\iota \circ \phi$ are non-conjugate while obviously the images of ι and $\iota \circ \phi$ are equal as subgroups. This is similar to the difference between the conjugacy classification of order ℓ elements and order ℓ subgroups in [14].*

Let $\phi \colon \mathrm{C}_\ell \to \mathrm{C}_\ell$ be an automorphism of the cyclic group. Then ϕ induces an $\mathcal{O}_{K,S}$-linear automorphism of $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$ in the obvious way. For the purposes of the next result, we call such automorphisms *special*.

Corollary 3.3. *The injection of Proposition 3.1 gives rise to a bijection from the conjugacy classes of order ℓ subgroups in $\mathrm{GL}_n(\mathcal{O}_{K,S})$ to the quotient of the set of isomorphism classes of $\mathcal{O}_{K,S}$-free $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules by the special automorphisms.*

Under the bijection of Proposition 3.1, the centralizer of a subgroup $C_\ell \hookrightarrow GL_n(\mathcal{O}_{K,S})$ is isomorphic to the $\mathcal{O}_{K,S}[C_\ell]$-automorphism group of the corresponding module M. The normalizer is isomorphic to the group of $\mathcal{O}_{K,S}$-automorphisms which are semilinear with respect to a special automorphism of $\mathcal{O}_{K,S}[C_\ell]$.

Proof. We consider a fixed subgroup (as opposed to a conjugacy class), and consider the associated module M, equipped with the corresponding choice of basis. Then a matrix A in the centralizer of $\iota\colon C_\ell \hookrightarrow GL_n(\mathcal{O}_{K,S})$ provides commutative diagrams for all $g \in C_\ell$:

$$
\begin{array}{ccc}
\mathcal{O}_{K,S}^{\oplus n} & \xrightarrow{\iota(g)} & \mathcal{O}_{K,S}^{\oplus n} \\
{\scriptstyle A}\Big\downarrow & & \Big\downarrow{\scriptstyle A} \\
\mathcal{O}_{K,S}^{\oplus n} & \xrightarrow[\iota(g)]{} & \mathcal{O}_{K,S}^{\oplus n}.
\end{array}
$$

As in the proof of Proposition 3.1, this provides an automorphism of the $\mathcal{O}_{K,S}[C_\ell]$-module M. This construction is obviously compatible with composition.

Conversely, an $\mathcal{O}_{K,S}[C_\ell]$-automorphism of the module M corresponding to $\iota(C_\ell)$ provides a change-of-basis matrix which is in the centralizer of $\iota(C_\ell)$. Again, this is obviously compatible with composition.

The two constructions above are inverses, proving the claim for the centralizer. The claims for the normalizer are proved in the same way, changing the lower morphism in the commutative diagram from $\iota(g)$ to $\phi \circ \iota(g)$. $\square$

Remark 3.4. *We will see later that the semi-linear automorphisms correspond to the action of the Galois group of $\mathcal{O}_{K,S}[\zeta_\ell]$ over $\mathcal{O}_{K,S}$ on $GL_n(\mathcal{O}_{K,S})$. Consequently, one of the contributions to the conjugacy classification is given by the Galois-orbits on the class group.*

4. MODULES OVER CYCLIC GROUP RINGS

In this section, we provide a generalization of Reiner's classification of $\mathbb{Z}[C_\ell]$-modules, cf. [15]. Reiner's analysis of the modules over the group ring $\mathbb{Z}[C_\ell]$ is essentially based on the class group theory for cyclotomic integers. In the generalization to rings of S-integers, we will therefore need some assumption on the situation, related to existence of relative integral bases.

4.1. **Relative integral bases.** As usual, denote by $\Phi_\ell(T)$ the ℓ-th cyclotomic polynomial. If $\Phi_\ell(T)$ is not K-irreducible, then the degree of ζ_ℓ over K is a strict divisor of the degree of $\Phi_\ell(T)$. In this case, we have $\mathcal{O}_{K,S}[\zeta_\ell] = \mathcal{O}_{K,S}[T]/(\Psi_\ell(T))$ where $\Psi_\ell(T)$ is the minimal polynomial of ζ_ℓ over K.

> *To get a full analogue of Reiner's result, we assume that the ring $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ is a Dedekind domain. Some results will work under the weaker hypothesis that $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain. We will make these cases explicit.*

Note that even if $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain, $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ need not be a Dedekind domain. If $\Phi_\ell(T)$ is not K-irreducible, then the total ring of fractions is $K[T]/(\Phi_\ell(T))$ which is a direct sum of copies of $K(\zeta_\ell)$, corresponding to the number of K-factors of $\Phi_\ell(T)$.

Example 4.1. *In the case $K = \mathbb{Q}(\sqrt{-7})$ and $\ell = 7$, denote by $N_7 = \sum_{i=0}^{6}[g^i]$ the norm element in $\mathbb{Z}[C_7]$, where $g \in C_7$ is a generator. Then $\mathcal{O}_K[C_\ell]/(N_7)$ is a fiber product of two copies of $\mathcal{O}_K[\zeta_7]$ over the quotient $\mathcal{O}_K[\zeta_7]/(\sqrt{-7}^3)$ where $\sqrt{-7}^3$ is the resultant of the two K-factors of $\Phi_7(T) = \Psi_7(T) \cdot \overline{\Psi_7(T)}$, where*

$$
\begin{aligned}
\Psi_7(T) &= T^3 - (\zeta_7 + \zeta_7^2 + \zeta_7^4)T^2 + (\zeta_7^3 + \zeta_7^5 + \zeta_7^6)T - 1 \\
&= T^3 + \left(\frac{1 - \sqrt{-7}}{2}\right)T^2 - \left(\frac{1 + \sqrt{-7}}{2}\right)T - 1
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\Psi_7(T)} &= T^3 - (\zeta_7^3 + \zeta_7^5 + \zeta_7^6)T^2 + (\zeta_7 + \zeta_7^2 + \zeta_7^4)T - 1 \\
&= T^3 + \left(\frac{1 + \sqrt{-7}}{2}\right)T^2 - \left(\frac{1 - \sqrt{-7}}{2}\right)T - 1.
\end{aligned}
$$

$\square$

The Dedekind domain requirement is crucial because it provides a bijection between isomorphism classes of finitely generated torsion-free modules of fixed rank n and the class group. The ring $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind ring precisely when the relevant powers of ζ_ℓ form a relative integral basis of $K(\zeta_\ell)/K$. For most of our purposes, the following statement will be sufficient.

Lemma 4.2. *Let $K/\mathbb{Q}$ be a Galois extension with discriminant d_K, and let ℓ be a prime with $(\ell, d_K) = 1$. Then*
$$\mathcal{O}_{K(\zeta_\ell)} = \mathcal{O}_K[\zeta_\ell] \cong \mathcal{O}_K[T]/(\Phi_\ell(T)).$$

Proof. The discriminant of $\mathbb{Q}(\zeta_\ell)/\mathbb{Q}$ is a power of ℓ so that by assumption the discriminants of K and $\mathbb{Q}(\zeta_\ell)$ are coprime. Then the product of the integral bases of $K/\mathbb{Q}$ and $\mathbb{Q}(\zeta_\ell)/\mathbb{Q}$ is an integral basis of $K(\zeta_\ell)/\mathbb{Q}$. In particular, any element of $\mathcal{O}_{K(\zeta_\ell)}$ is an $\mathcal{O}_K$-linear combination of $1, \zeta_\ell, \ldots, \zeta_\ell^{\ell-1}$, hence these form a relative integral basis of $K(\zeta_\ell)/K$. $\qquad\square$

Remark 4.3. *For the case $K = \mathbb{Q}(\sqrt{-m})$ and $3 \mid m$, then there doesn't exist a relative integral basis for the extension $\mathbb{Q}(\sqrt{-m}, \zeta_3)/\mathbb{Q}(\sqrt{-m})$. In particular, the ring $\mathcal{O}_{\sqrt{-m}}[\zeta_3]$ isn't the maximal order of $\mathbb{Q}(\sqrt{-m}, \zeta_3)$ and hence fails to be a Dedekind ring. This follows from [2, Theorem I].*

4.2. **Isomorphism classification.** We can now provide our extension of Reiner's study of isomorphism classes of modules over group rings for cyclic groups, mostly following the arguments of [15]. The situation is the following: let K be a number field, let S be a finite set of places containing the infinite ones, and denote by $\mathcal{O}_{K,S}$ the ring of S-integers in K. Denote by C_ℓ the cyclic group of order ℓ where ℓ is a prime. In some cases relevant to computations of Farrell–Tate cohomology, we will give a classification of finitely generated $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules which are $\mathcal{O}_{K,S}$-free. For this, we assume that $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ is a Dedekind domain. Note that in this case we actually have $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T)) \cong \mathcal{O}_{K,S}[\zeta_\ell]$. Denote by $\mathrm{N} = \sum_{g \in \mathrm{C}_\ell}[g]$ the norm element.

Theorem 4.4. *Let $\mathcal{O}_{K,S}$ be a ring of S-integers in a number field and let ℓ be a prime. Assume that $\mathcal{O}_{K,S}$ is a principal ideal domain and that $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ is a Dedekind domain. Then the isomorphism classes of finitely generated $\mathcal{O}_{K,S}$-free $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules M are parametrized by*

   (1) the $\mathcal{O}_{K,S}[\zeta_\ell]$-rank r of $M_\mathrm{N} := \{m \in M \mid \mathrm{N} \cdot m = 0\}$,
   (2) the ideal class of the determinant of the $\mathcal{O}_{K,S}[\zeta_\ell]$-module M_N,
   (3) the $\mathcal{O}_{K,S}$-rank s of M/M_N,
   (4) a $\min(r, s)$-tuple of $\mathcal{O}_{K,S}[\zeta_\ell]^\times$-orbits of elements in $\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$ (for the natural multiplication action).

In the above, any non-negative integer is possible in (1) and (3). Any ideal class in (2) is possible, and any choice of tuple of orbits in (4) is possible.

Proof. Choose a generator γ for the cyclic group C_ℓ, so to write $\mathrm{C}_\ell = \langle \gamma \mid \gamma^\ell = 1 \rangle$. Let M be an $\mathcal{O}_{K,S}$-free $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-module. The set M_N of elements of M annihilated by the norm element N has a natural module structure over the quotient ring $\mathcal{O}_{K,S}[\mathrm{C}_\ell]/(\mathrm{N})$. The kernel of the natural surjective morphism
$$\mathcal{O}_{K,S}[\mathrm{C}_\ell] \to \mathcal{O}_{K,S}[T]/(\Phi_\ell(T)) : \gamma \mapsto T$$
is generated by $\Phi_\ell(\gamma) = \mathrm{N}$. In particular, we get an induced isomorphism
$$\mathcal{O}_{K,S}[\mathrm{C}_\ell]/(\mathrm{N}) \cong \mathcal{O}_{K,S}[T]/(\Phi_\ell(T)) \cong \mathcal{O}_{K,S}[\zeta_\ell].$$

Since M is $\mathcal{O}_{K,S}$-free, the submodule M_N embeds into a direct sum of copies of $K(\zeta_\ell)$ and hence is finitely generated and torsion-free over $\mathcal{O}_{K,S}[\zeta_\ell]$. By assumption, $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind ring, hence finitely generated and torsion-free implies projective and the general theory states that M_N is of the form $\mathcal{O}_{K,S}[\zeta_\ell]^{r-1} \oplus \mathfrak{a}$ with $\mathfrak{a}$ a fractional ideal of $\mathcal{O}_{K,S}[\zeta_\ell]$. The $\mathcal{O}_{K,S}[\zeta_\ell]$-module (and the restricted $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-module) M_N is completely determined by r and the ideal class of $\mathfrak{a}$. This provides the data in (1) and (2).

There is an inclusion of $\mathcal{O}_{K,S}[\zeta_\ell]$-modules $M_\mathrm{N} \supset (\gamma - 1)M \supset (\zeta_\ell - 1)M_\mathrm{N}$. From standard results on Dedekind rings (as in Reiner's paper), we have
$$M_\mathrm{N} \cong \mathcal{O}_{K,S}[\zeta_\ell]^{r-1} \oplus \mathfrak{a} \text{ (as } \mathcal{O}_{K,S}[\zeta_\ell]\text{-modules)};$$
and in this module, $(\gamma - 1)M$ is of the form $I_1 \oplus \cdots \oplus I_{r-1} \oplus I_r\mathfrak{a}$ for ideals $(\zeta_\ell - 1) \subseteq I_j \subseteq \mathcal{O}_{K,S}[\zeta_\ell]$. Hence the quotient $B := (\gamma - 1)M/(\zeta_\ell - 1)M_\mathrm{N}$ is a module over the quotient ring $\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$, and the latter ring is isomorphic to $\mathcal{O}_{K,S}/(\ell)$ because the same is true over $\mathbb{Z}$. Consequently, the module $(\gamma - 1)M$ is determined exactly by an ideal in $(\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1))^r \cong (\mathcal{O}_{K,S}/(\ell))^r$.

Remark 4.5. *In Reiner's work [15], we have $\mathbb{Z}/(\ell) \cong \mathbb{F}_\ell$ and the ideal is simply determined by an integer $\leq r$. The same is still true whenever the prime ℓ is inert in the field extension $K/\mathbb{Q}$.*

The quotient M/M_N is a finitely generated torsion-free $\mathcal{O}_{K,S}$-module: by assumption M embeds into a direct sum of copies of K and $K(\zeta_\ell)$ and M_N is the part of M which embeds into the $K(\zeta_\ell)$-copies. Hence it is projective and the sequence $0 \to M_N \to M \to M/M_N \to 0$ splits as $\mathcal{O}_{K,S}$-modules. The module M is $\mathcal{O}_{K,S}$-free by assumption. Therefore, as $\mathcal{O}_{K,S}$-modules, we have $M_N \cong \mathcal{O}_{K,S}^a \oplus \mathfrak{b}$ and $M/M_N \cong \mathcal{O}_{K,S}^b \oplus \mathfrak{b}^{-1}$ for some fractional $\mathcal{O}_{K,S}$-ideal $\mathfrak{b}$. The module M/M_N (both as $\mathcal{O}_{K,S}$-module and as $\mathcal{O}_{K,S}[C_\ell]$-module) is determined up to isomorphism by b and the ideal class of $\mathfrak{b}$. Since the ideal $\mathfrak{b}$ is equivalent to the norm of the ideal $\mathfrak{a}$ in the extension $\mathcal{O}_{K,S}[\zeta_\ell]/\mathcal{O}_{K,S}$, its ideal class is determined by the one of $\mathfrak{a}$. This provides the information in (3).

It remains to identify the $\mathcal{O}_{K,S}[C_\ell]$-module structure of M in terms of the module structures of M_N and M/M_N. The module M is an extension $0 \to M_N \to M \to M/M_N \to 0$ where M_N has the $\mathcal{O}_{K,S}[C_\ell]$-structure induced from the module structure over $\mathcal{O}_{K,S}[\zeta_\ell]$ with a non-trivial ζ_ℓ-action, and the structure of M/M_N is induced from $\mathcal{O}_{K,S}$, i.e., has a trivial ζ_ℓ-action. We noted above that M/M_N is $\mathcal{O}_{K,S}$-projective, so that the extension splits as $\mathcal{O}_{K,S}$-module. Hence we can, as in Reiner's paper, choose an $\mathcal{O}_{K,S}$-complement X of M_N lifting M/M_N and write $M = M_N \oplus X$. To write down the action of γ in this decomposition, we note that γ acts as ζ_ℓ on M_N and acts trivially on M/M_N. Therefore, we must have $\gamma(x) = x + (\gamma-1)(x)$ for an $\mathcal{O}_{K,S}$-linear map $(\gamma-1) \colon M/M_N \to M_N$. From the decomposition $M = M_N \oplus X$, we have

$$(\gamma - 1)M = (\zeta_\ell - 1)M_N \oplus (\gamma - 1)X.$$

Therefore, the map $(\gamma-1) \colon X \to M_N$ is determined, modulo $(\zeta_\ell - 1)M_N$, by an $\mathcal{O}_{K,S}$-linear map $X \to B$. To describe the map $X \to B$, we recall that $X \cong \mathcal{O}_{K,S}^b \oplus \mathfrak{b}^{-1}$ and B is a submodule of $\mathcal{O}_{K,S}/(\ell)^{\oplus r}$. Then $X \to B$ factors through a map $\mathcal{O}_{K,S}/(\ell)^{\oplus(b+1)} \to \mathcal{O}_{K,S}/(\ell)^{\oplus r}$, and as in Reiner's paper, by choosing appropriate bases, y_i of $\mathcal{O}_{K,S}/(\ell)^{\oplus(b+1)}$ and β_j of $\mathcal{O}_{K,S}/(\ell)^{\oplus r}$, the map $(\gamma - 1)$ has the form $y_j \mapsto c_j\beta_j$ for suitable coefficients c_i.

It remains to figure out which $\mathcal{O}_{K,S}/(\ell)$-multiples in the above give rise to isomorphic module structures. In Reiner's paper, this is taken care of by [15, Lemma 4]: over $\mathcal{O}_{K,S} = \mathbb{Z}$ all the coefficients can be taken to be 1. In our more general case, the appropriate replacement of [15, Lemma 4] is the following statement: for a fractional $\mathcal{O}_{K,S}[\zeta_\ell]$-ideal $\mathfrak{a}$, an element $\beta \in \mathfrak{a}$ and two elements $c_1, c_2 \in \mathcal{O}_{K,S}$, the two $\mathcal{O}_{K,S}[C_\ell]$-module structures on $\mathfrak{a} \oplus \mathcal{O}_{K,S} \cdot y$ given by $y \mapsto y + c_1\beta$ and $y \mapsto y + c_2\beta$, respectively, are isomorphic if and only if there exists a unit $u \in \mathcal{O}_{K,S}[\zeta_\ell]^\times$ such that $uc_1 = c_2$. In particular, it is in general not possible to have all the coefficients be 1, but coefficients in the same orbit of the unit group give rise to isomorphic actions. The data in (5) for the above action is therefore given by the orbits of the coefficients $c_i \in \mathcal{O}_{K,S}/(\ell) \cong \mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$ under the multiplication action of the unit group $\mathcal{O}_{K,S}[\zeta_\ell]^\times$.

The explicit construction of a module with the specified invariants goes through as in [15], showing the realizability of all choices. $\square$

Remark 4.6. *It should be pointed out that the assumption that $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ is a Dedekind ring is important. What comes up naturally in the first step are finitely generated torsion-free modules. That classification problem is fairly complicated, but over Dedekind rings reduces to the classification of finitely generated locally free modules.*

We now formulate a very special case of the classification which works under the weaker assumption that $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind ring but in which $\Phi_\ell(T)$ need not be K-irreducible. We restrict to the case where M_N has $\mathcal{O}_{K,S}[\zeta_\ell]$-rank 1. In this case, base-change to K results in one of the irreducible K-representations of C_ℓ. The $\mathcal{O}_{K,S}[T]/(\Phi_\ell)$-module structure factors through a projection $\mathcal{O}_{K,S}[T]/(\Phi_\ell) \twoheadrightarrow \mathcal{O}_{K,S}[\zeta_\ell]$ and is completely determined by this. Again, the $\mathcal{O}_{K,S}[C_\ell]$-module structure of M_N is completely determined by a fractional ideal $\mathfrak{a}$ in $\mathcal{O}_{K,S}[\zeta_\ell]$. The rest of the analysis goes through to show the following

Proposition 4.7. *Let $\mathcal{O}_{K,S}$ be a ring of S-integers in a number field K and let ℓ be a prime. Assume that $\mathcal{O}_{K,S}$ is a principal ideal domain and that $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain. Then the isomorphism classes of finitely generated $\mathcal{O}_{K,S}$-free $\mathcal{O}_{K,S}[C_\ell]$-modules M where M_N has $\mathcal{O}_{K,S}[\zeta_\ell]$-rank 1 are parametrized by*

(1) the ideal class of the $\mathcal{O}_{K,S}[\zeta_\ell]$-module M_N,
(2) the $\mathcal{O}_{K,S}$-rank of M/M_N,
(3) an $\mathcal{O}_{K,S}[\zeta_\ell]^\times$-orbit of elements in $\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$ (for the natural multiplication action).

In the above, any integer $n \geq 0$ is possible in (2). Any ideal class is possible in (1), and any choice of orbit in (4) is possible.

Remark 4.8. *Pulling back modules along the two projections*

$$\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}[T]/(\Phi_7(T)) \to \mathcal{O}_{\mathbb{Q}(\zeta_7)}$$

results in non-isomorphic modules which correspond to non-isomorphic $\mathbb{Q}(\sqrt{-7})$-representations of C_7. However, this effectively only amounts to different choices of generators of conjugate subgroups. If we are only interested in counting subgroups, this doesn't affect the end result.

5. Centralizers and normalizers

We now need to describe centralizers and normalizers of the corresponding C_ℓ-subgroups of $\mathrm{GL}_n(\mathcal{O}_{K,S})$. For the purpose of the following section, fix a subgroup $\iota\colon \mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ and the corresponding $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-module M. Since our intended application is to essential rank one cases, most notably $\mathrm{GL}_3(\mathcal{O}_{K,S})$, we assume throughout the section that the associated module M is such that its associated representation over K is of the form $K \times K(\zeta_\ell)$. We also assume in the following section that the conditions of Proposition 4.7 are satisfied.

First, we can embed $\mathrm{GL}_n(\mathcal{O}_{K,S}) \hookrightarrow \mathrm{GL}_n(K)$. The centralizer of $\mathrm{C}_\ell \hookrightarrow \mathrm{GL}_n(K)$ is the automorphism group of the representation $M \otimes_{\mathcal{O}_{K,S}} K \cong K \times K(\zeta_\ell)$ of C_ℓ over K. Under our assumption $\zeta_\ell \notin K$, the C_ℓ-representation $K(\zeta_\ell)$ is K-irreducible. In particular,

$$\mathrm{Hom}_{K[\mathrm{C}_\ell]}(K(\zeta_\ell), K) \cong \mathrm{Hom}_{K[\mathrm{C}_\ell]}(K, K(\zeta_\ell)) \cong 0.$$

From this, any $K[\mathrm{C}_\ell]$-automorphism ϕ of $K \times K(\zeta_\ell)$ must be of the form $\phi_K \times \phi_{K(\zeta_\ell)}$ where ϕ_K and $\phi_{K(\zeta_\ell)}$ are $K[\mathrm{C}_\ell]$-automorphisms of K and $K(\zeta_\ell)$, respectively. Via the embedding $\mathrm{GL}_n(\mathcal{O}_{K,S}) \hookrightarrow \mathrm{GL}_n(K)$, the same must be true for automorphisms of the $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-modules. In terms of the centralizer as a subgroup of $\mathrm{GL}_n(\mathcal{O}_{K,S})$, this means that the centralizer must be conjugate to a block-diagonal matrix. For the normalizer, similar statements apply. The only additional elements in the normalizer would come from K-linear automorphisms of $K(\zeta_\ell)$ which are accounted for by the Galois group $\mathrm{Gal}(K(\zeta_\ell)/K)$.

Now we need some induction-type theorems to determine the automorphism groups of the individual almost-direct summands of the module M.

Lemma 5.1. *Let M be an $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-module such that multiplication with the norm element N is the zero map and assume that the $\mathcal{O}_{K,S}[\zeta_\ell]$-rank of M is 1. Then*

$$\mathrm{Aut}_{\mathcal{O}_{K,S}[\mathrm{C}_\ell]}(M) \cong \mathrm{Aut}_{\mathcal{O}_{K,S}[\zeta_\ell]}(M) \cong \mathcal{O}_{K,S}[\zeta_\ell]^\times.$$

Proof. Since the norm element N annihilates M, it has an induced module structure for

$$\mathcal{O}_{K,S}[\mathrm{C}_\ell]/(\mathrm{N}) \cong \mathcal{O}_{K,S}[\zeta_\ell].$$

This yields a homomorphism $\mathrm{Aut}_{\mathcal{O}_{K,S}[\mathrm{C}_\ell]}(M) \to \mathrm{Aut}_{\mathcal{O}_{K,S}[\zeta_\ell]}(M)$. This homomorphism is injective, since both automorphism groups embed into $\mathrm{Aut}_{\mathcal{O}_{K,S}}(M)$. The natural restriction map along the homomorphism $\mathcal{O}_{K,S}[\mathrm{C}_\ell] \to \mathcal{O}_{K,S}[\zeta_\ell]$ provides an inverse, establishing the first isomorphism.

For the second isomorphism, we know that M is a finitely generated projective $\mathcal{O}_{K,S}[\zeta_\ell]$-module, and our additional assumption is that its rank is 1. Since local units can be patched to global units, the automorphism group of a finitely generated projective $\mathcal{O}_{K,S}[\zeta_\ell]$-module of rank 1 is isomorphic to $\mathcal{O}_{K,S}[\zeta_\ell]^\times$. $\qquad\square$

Lemma 5.2. *Let M be an $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-module such that multiplication with the norm element N is injective and M is a finitely generated projective $\mathcal{O}_{K,S}$-module of rank 1. Then*

$$\mathrm{Aut}_{\mathcal{O}_{K,S}[\mathrm{C}_\ell]}(M) \cong \mathrm{Aut}_{\mathcal{O}_{K,S}}(M) \cong \mathcal{O}_{K,S}^\times.$$

Proof. Injectivity of multiplication with the norm implies that the action of C_ℓ is trivial. The second isomorphism $\mathrm{Aut}_{\mathcal{O}_{K,S}}(M) \cong \mathcal{O}_{K,S}^\times$ follows as before, by patching local units.

An $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-automorphism of M is in particular an $\mathcal{O}_{K,S}$-automorphism, giving rise to an injective restriction map $\mathrm{Aut}_{\mathcal{O}_{K,S}[\mathrm{C}_\ell]}(M) \to \mathrm{Aut}_{\mathcal{O}_{K,S}}(M)$. Since any $\mathcal{O}_{K,S}$-automorphism of M automatically commutes with the trivial C_ℓ-action we get the first isomorphism. $\qquad\square$

We can now analyse the structure of $\mathrm{Aut}_{\mathcal{O}_{K,S}[\mathrm{C}_\ell]}(M)$ where M is a module corresponding to a C_ℓ-subgroup of $\mathrm{GL}_3(\mathcal{O}_{K,S})$. Any $\mathcal{O}_{K,S}[\mathrm{C}_\ell]$-automorphism ϕ of M necessarily maps the submodule M_N to itself and hence induces automorphisms ϕ_N and $\overline{\phi}$ of M_N and M/M_N, respectively. By Lemmas 5.1 and 5.2, we have an induced morphism

$$\mathrm{Aut}_{\mathcal{O}_{K,S}[\mathrm{C}_\ell]}(M) \to \mathcal{O}_{K,S}[\mathrm{C}_\ell]^\times \times \mathcal{O}_{K,S}^\times.$$

For the split module (corresponding to the orbit of 0 in $\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$ as described in Theorem 4.4 resp. Proposition 4.7), this actually describes the full centralizer. For a non-split module where there is an additional unipotent action (corresponding to the orbit of a non-zero element in in $\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$), we have morphisms $\mathcal{O}_{K,S}[\zeta_\ell] \to \mathcal{O}_{K,S}/(\ell)$ and $\mathcal{O}_{K,S} \to \mathcal{O}_{K,S}/(\ell)$ given by reduction mod ℓ. These ring homomorphisms induce maps on the unit groups.

Lemma 5.3. *Assume M is the $\mathcal{O}_{K,S}[C_\ell]$-module associated to a C_ℓ-subgroup of $\mathrm{GL}_3(\mathcal{O}_{K,S})$ where $[K(\zeta_\ell) : K] = 2$. The induced morphism from the automorphism group above factors through an isomorphism*

$$\mathrm{Aut}_{\mathcal{O}_{K,S}[C_\ell]}(M) \to \mathcal{O}_{K,S}[C_\ell]^\times \times^c_{\mathrm{End}(\mathcal{O}_{K,S}/(\ell))} \mathcal{O}_{K,S}^\times$$

where the right-hand side is defined to be

$$\mathcal{O}_{K,S}[C_\ell]^\times \times^c_{\mathrm{End}(\mathcal{O}_{K,S}/(\ell))} \mathcal{O}_{K,S}^\times = \left\{ (\phi, \psi) \mid \overline{\phi}c = c\overline{\psi} \ in \ \mathrm{End}(\mathcal{O}_{K,S}/(\ell)) \right\}$$

Proof. It remains to identify the image of the induced morphism. Let

$$(\phi, \psi) \in \mathcal{O}_{K,S}[C_\ell]^\times \times^c_{\mathrm{End}(\mathcal{O}_{K,S}/(\ell))} \mathcal{O}_{K,S}^\times.$$

To set up notation, let $M = \mathfrak{a} \oplus \mathrm{Nm}\,\mathfrak{a}^{-1}$ with the action specified as in Reiner's results: it sends a generator y to (β, y) where $\beta \in \mathfrak{a}$ is a choice of preimage of an element $c \in \mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$. Formulated differently, the action on $x \in \mathrm{Nm}\,\mathfrak{a}^{-1}$ adds a specific choice of lift $\tilde{\overline{x}} \in \mathfrak{a}$ of the product $c\overline{x}$ of the reduction of x mod ℓ and the coefficient c; for notational purposes, we denote this lift $\tilde{\overline{x}}$ by $\beta(x)$.

Now we want to determine when the action described above commutes with the automorphism (ϕ, ψ). If we first apply the action and then the automorphism, then we get $\phi(\beta(y))$ in the component $\mathfrak{a}$. If, on the other hand, we first apply the automorphism and then the action, we get $\beta(\psi(y))$ in the component $\mathfrak{a}$. For $\phi(\beta(y)) = \beta(\psi(y))$, it is necessary and sufficient that the reductions of ϕ and ψ to $\mathcal{O}_{K,S}/(\ell)$ satisfy $\overline{\phi}(c \cdot \overline{y}) = c \cdot \overline{\psi}(\overline{y})$. This is precisely the claim. $\qquad\square$

Remark 5.4. *Since $\overline{\phi}$ and $\overline{\psi}$ are given by multiplication with units in $\mathcal{O}_{K,S}/(\ell)$ (the reductions of the respective units in the rings of integers), they are $\mathcal{O}_{K,S}/(\ell)$-linear. In particular, the requirement translates to $c(\overline{\phi} - \overline{\psi})$, and this is always satisfied if the reductions of ϕ and ψ are the same. However, if c happens to be a zero-divisor, the requirement $c(\overline{\phi} - \overline{\psi})$ is strictly weaker; it is no requirement at all for $c = 0$, which recovers the product description of the automorphism group for the split module.*

Lemma 5.5. *Assume M is the $\mathcal{O}_{K,S}[C_\ell]$-module associated to a C_ℓ-subgroup of $\mathrm{GL}_3(\mathcal{O}_{K,S})$ with $[K(\zeta_\ell) : K] = 2$. In particular, $M \cong \mathfrak{a} \oplus \mathrm{Nm}\,\mathfrak{a}^{-1}$ for an ideal class $\mathfrak{a}$ of $\mathcal{O}_{K,S}[\zeta_\ell]$. The group of special semilinear automorphisms of M is of the form*

$$\left(\mathrm{Aut}_{\mathcal{O}_{K,S}[C_\ell]}(M) \right) \rtimes \mathrm{Stab}(\mathfrak{a}, \mathrm{Gal}(K(\zeta_\ell)/K)).$$

The action is the natural Galois action on the automorphism group, viewed as fiber product of unit groups as in Lemma 5.3.

Proof. By embedding $\mathcal{O}_{K,S}[C_\ell] \hookrightarrow K[C_\ell]$, we already know that the only semilinear automorphisms that are not in the automorphism group come from the Galois-action of $\mathrm{Gal}(K(\zeta_\ell)/K)$. However, the Galois group does not need to stabilize the isomorphism class of the module; this happens whenever we have a non-trivial Galois action on the class group of $K(\zeta_\ell)$. The semilinear automorphisms modulo the linear ones are exactly identified with the stabilizer of the ideal class $\mathfrak{a}$ in the Galois group, as claimed. $\qquad\square$

6. Example cases

Now we discuss a couple of example cases to compare them to the computer calculations as sanity check.

6.1. **Homological 3-torsion in PGL_3 over quadratic imaginary integers.** Let m be a square-free natural number with $3 \nmid m$ and denote by $\mathcal{O}_{-m} = \mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$. In the case where $\mathcal{O}_{-m}$ is a principal ideal domain, using Corollary 3.3 combined with Proposition 4.7, the conjugacy classes of embeddings $C_3 \hookrightarrow \mathrm{PGL}_3(\mathcal{O}_{-m})$ are parametrized by pairs of elements $(\mathfrak{a}, c)$ where $\mathfrak{a}$ is an ideal class in $\mathcal{O}_{-m}[\zeta_3]$ and c is an $\mathcal{O}_{-m}[\zeta_3]^\times$-orbit of elements in $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1)$. Consequently, the number of conjugacy classes of subgroups is

$$\frac{\# \mathrm{Cl}_{\mathbb{Q}(\sqrt{-d}, \sqrt{-3})}}{\# \mathrm{Gal}(\mathbb{Q}(\sqrt{-d}, \sqrt{-3})/\mathbb{Q}(\sqrt{-d}))} \cdot \# \left((\mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1))/\mathcal{O}_{-m}[\zeta_3]^\times \right)$$

To determine the orbit set for the natural $\mathcal{O}_{-m}[\zeta_3]^\times$-action on $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1)$ we first note that our assumption $3 \nmid m$ implies

$$\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}/(3) \cong \mathbb{F}_3[X]/(X^2 + m) \cong \begin{cases} \mathbb{F}_9 & m \equiv 1 \bmod 3 \\ \mathbb{F}_3 \times \mathbb{F}_3 & m \equiv 2 \bmod 3 \end{cases}$$

We make a case distinction, depending on the residue class of m mod 3. These arguments actually do not require $\mathcal{O}_{-m}$ to be a principal ideal domain. We do them in this generality: for any m coprime to 3,

the numbers of orbits below give the numbers of conjugacy classes of order-3-subgroups corresponding to the trivial ideal class.

Proposition 6.1. *Assume $m \equiv 2 \bmod 3$. If the reduction morphism $\mathcal{O}_{-m}[\zeta_3]^\times \to (\mathbb{F}_3 \times \mathbb{F}_3)^\times$ is surjective, then there are four $\mathcal{O}_{-m}[\zeta_3]^\times$-orbits on $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1)$. Otherwise, the reduction morphism has image $\{(1,1),(-1,-1)\}$ and there are five orbits.*

Proof. If $m \equiv 2 \bmod 3$, then $(\mathbb{F}_3 \times \mathbb{F}_3)^\times \cong \mathbb{Z}/2\mathbb{Z}^{\times 2}$, concretely realized as the subset $\{(\pm 1, \pm 1)\}$. Note that actually $\mathbb{F}_3 \times \mathbb{F}_3 \cong \mathbb{F}_3[X]/(X^2 + m)$, so that an integer $n \in \mathbb{Z}$ always reduces to $(\bar{n}, \bar{n})$, and $\sqrt{-m} \in \mathcal{O}_{-m}$ maps to $(1, -1)$. In particular, if we consider the natural reduction morphism $\mathcal{O}_{-m}[\zeta_3]^\times \to (\mathbb{F}_3 \times \mathbb{F}_3)^\times$, the elements $(1,1)$ and $(-1,-1)$ are the images of the global units $\pm 1 \in \mathcal{O}_{-m}[\zeta_3]$. So there are only two possibilities for the image of the reduction map: either it is $\mathbb{Z}/2\mathbb{Z} \cong \{(1,1),(-1,-1)\}$ or it is the full group $(\mathbb{F}_3 \times \mathbb{F}_3)^\times$. Consequently, there are two possibilities for the orbit set. In both of them, we have the orbits $\{(0,0)\}$, $\{(1,0),(-1,0)\}$ and $\{(0,1),(0,-1)\}$, irrespective of the image. If the image contains $(1,-1)$, then we have an orbit $\{(\pm 1, \pm 1)\}$. If the image doesn't contain $(1,-1)$, then we have two orbits $\{(1,1),(-1,-1)\}$ and $\{(1,-1),(-1,1)\}$. This proves the claim. $\square$

Example 6.2. *We give examples that both possibilities can appear.*
Consider the case $m = 2$. Using Pari/GP, we find that $\zeta_3 - \sqrt{-2}\zeta_3 + 2$ is a fundamental unit of $\mathcal{O}_{-2}[\zeta_3]$. Using that integers go to their reduction, $\sqrt{-2}$ maps to $(1,2)$ and ζ_3 maps to $(1,1)$, the image of this unit under the reduction map is $(1,1) - (1,2)(1,1) + (2,2) = (2,1)$. Thus we are in the case where there are 4 orbits.
Consider the case $m = 5$. Using Pari/GP, we find that $-\sqrt{-5} + 4(\zeta_3 + 1) + \zeta_3\sqrt{-5}$ is a fundamental unit of $\mathcal{O}_{-5}[\zeta_3]$. The image of this unit under the reduction map is $(2,1) + (8,8) + (1,2) = (-1,-1)$. In particular, $(1,2)$ is not in the image of the reduction map and therefore we are in the case with 5 orbits. $\square$

Example 6.3. *For the first few square-free numbers m with $m \equiv 2 \bmod 3$ and $-m \equiv 2, 3 \bmod 4$, we then have the following*

m	$h_{\mathbb{Q}(\sqrt{-m}, \sqrt{-3})}$	*orbit number*
2	1	4
5	2	5
14	4	5
17	4	5
26	12	4
29	6	5

It seems that there is a relation to the unit index in the class number formula, cf. [10]. In the above table, the orbit number 4 appears precisely in those cases where $h_{\mathbb{Q}(\sqrt{-m}, \sqrt{-3})} = h_{\mathbb{Q}(\sqrt{-m})} h_{\mathbb{Q}(\sqrt{3m})}$; and the orbit number 5 appears in those cases where $h_{\mathbb{Q}(\sqrt{-m}, \sqrt{-3})} = (1/2) h_{\mathbb{Q}(\sqrt{-m})} h_{\mathbb{Q}(\sqrt{3m})}$. $\square$

Proposition 6.4. *Assume $m \equiv 1 \bmod 3$. The number of $\mathcal{O}_{-m}[\zeta_3]^\times$-orbits is given by the following table, depending on the image of the natural reduction morphism $\mathcal{O}_{-m}[\zeta_3]^\times \to (\mathbb{F}_9)^\times \cong \mathbb{Z}/8\mathbb{Z}$:*

image	*orbit number*
$\mathbb{Z}/8\mathbb{Z}$	2
$\mathbb{Z}/4\mathbb{Z}$	3
$\mathbb{Z}/2\mathbb{Z}$	5

Proof. The subgroup $\{\pm 1\}$ is always contained in the image of the global units $\pm 1 \in \mathcal{O}_{-m}[\zeta_3]$, hence the above list exhausts all possible cases. The orbits are then given by $\{0\}$ and the cosets of the image of the reduction map in $\mathbb{F}_9^\times \cong \mathbb{Z}/8\mathbb{Z}$. $\square$

Example 6.5. *Again, all three cases appear as can be checked by computing fundamental units using Pari/GP. This is most easily done in the cases where $-m \equiv 2, 3 \bmod 4$ since in this case $\zeta_3\sqrt{-m}$ is a primitive element giving rise to the integral basis $\{1, \zeta_3 + 1, \zeta_3\sqrt{-m}, -\sqrt{-m}\}$. Once the fundamental unit is computed, its image under the reduction map is determined by noting that integers $n \in \mathbb{Z}$ map to the subfield $\mathbb{F}_3 \subset \mathbb{F}_9$ via reduction mod 3, ζ_3 maps to 1 and $\sqrt{-m}$ maps to a primitive element of the field extension $\mathbb{F}_9/\mathbb{F}_3$. If the fundamental unit maps to the reduction of an integer, the image is $\mathbb{Z}/2\mathbb{Z}$, if it maps to $\pm\sqrt{-m}$ the image is $\mathbb{Z}/4\mathbb{Z}$ and if it maps to a linear combination $\pm 1 \pm \sqrt{-m}$ we have the full image. For the first few square-free m, the results look as follows:*

m	$h_{\mathbb{Q}(\sqrt{-m},\sqrt{-3})}$	orbit number
1	1	2
10	2	4
13	4	3
22	2	4

Again, there seems to be a relation to the unit index. It seems reasonable to expect that orbit number 2 only appears in the exceptional case $m = 1$ where the unit group of $\mathcal{O}_{-m}$ is μ_4; orbit number 3 should appear in those cases where $h_{\mathbb{Q}(\sqrt{-m},\sqrt{-3})} = h_{\mathbb{Q}(\sqrt{-m})}h_{\mathbb{Q}(\sqrt{3m})}$; and the orbit number 4 should appear in those cases where $h_{\mathbb{Q}(\sqrt{-m},\sqrt{-3})} = (1/2)h_{\mathbb{Q}(\sqrt{-m})}h_{\mathbb{Q}(\sqrt{3m})}$. $\qquad\square$

Now it remains to determine the structure of the normalizers and compute the appropriate contributions to the Farrell–Tate cohomology. The corresponding centralizers will be of the form

$$\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times^c_{\mathrm{End}(\mathcal{O}_{-m}/(3))} \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m})}$$

for an $\mathcal{O}_{-m}[\zeta_3]^{\times}$-orbit representative $c \in \mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1)$. For the case where $m \equiv 1 \bmod 3$, c is either 0 or a unit, giving rise to the two possibilities

$$\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m})} \quad \text{and} \quad \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times_{\left(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}/(3)\right)^{\times}} \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m})},$$

respectively. In the case $m \equiv 2 \bmod 3$, there are two further possibilities for the orbits $\{(\pm 1, 0)\}$ and $\{(0, \pm 1)\}$. In these cases,

$$\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times^c_{\mathrm{End}(\mathcal{O}_{-m}/(3))} \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m})}$$

consists of those pairs $(\phi, \psi) \in \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m})}$ for which the reductions of ϕ and ψ agree in the first or second component of $(\mathbb{F}_3 \times \mathbb{F}_3)^{\times} \cong \mathbb{Z}/2\mathbb{Z}^{\times 2}$, respectively. The normalizers (whenever they do not already agree with the centralizers) will be extensions of the above groups by the group

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{-m},\zeta_3)/\mathbb{Q}(\sqrt{-m})) \cong \mathbb{Z}/2\mathbb{Z}$$

acting via the Galois action $\zeta_3 \mapsto \zeta_3^2$ on the first factor and trivially on the second. Note that these actions are actually compatible via the reduction to $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1) \cong \mathcal{O}_{-m}/(3)$ because the extension $\mathbb{Q}(\sqrt{-m},\zeta_3)/\mathbb{Q}(\sqrt{-m})$ is completely ramified over (3).

By Dirichlet's unit theorem, we have

$$\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \cong \mathbb{Z} \times \mu_{3n}, \text{ and } \mathcal{O}^{\times}_{-m} \cong \mu_n$$

where $n = 2$ except in the case $m = 1$ where $n = 4$. The Galois action on μ_n is trivial because these are contained in $\mathcal{O}_{-m}$; and the Galois action on $\mathbb{Z}$ must also be non-trivial, i.e., given by multiplication with -1, since none of the non-torsion units is contained in $\mathcal{O}_{-m}$. The structure of the centralizer for the split representation is therefore

$$\left(\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m})}\right) \rtimes \mathbb{Z}/2\mathbb{Z} \cong ((\mathbb{Z} \times \mu_3) \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mu_n^{\times 2}$$

with n as above.

For the normalizers of the non-split representations, i.e., where the orbit in $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3-1)$ is different from $\{0\}$ we can again consider the two cases:

(1) If $m \equiv 2 \bmod 3$, we have $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3-1) \cong \mathbb{F}_3 \times \mathbb{F}_3$. The reduction map $\mu_2 \cong \mathcal{O}^{\times}_{-m} \to (\mathbb{F}_3 \times \mathbb{F}_3)^{\times}$ is injective. For $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \cong \mathbb{Z} \times \mu_{3n}$, the reduction map $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \to (\mathbb{F}_3 \times \mathbb{F}_3)^{\times}$ is injective on μ_n, the zero map on μ_3 and the image of a fundamental unit depends on m. In the case where $c = \{(\pm 1, 0)\}$ or $c = \{(0, \pm 1)\}$, the fiber product is given by the units in $\mathcal{O}_{-m}[\zeta_3]$ whose reduction has first resp. second component equal to the reduction of a unit from $\mathcal{O}_{-m}$, respectively. Since this is no condition at all, the fiber product is simply $\mathcal{O}_{-m}[\zeta_3]^{\times} \cong \mathbb{Z} \times \mu_{3n}$. In the case where the orbit contains $(1, -1)$, the fiber product consists of the group of units in $\mathcal{O}_{-m}[\zeta_3]$ whose reduction is of the form $(1, 1)$ or $(-1, -1)$. The index of this subgroup is 1 or 2, depending on m, but in any case the isomorphism type of the units is again $\mathbb{Z} \times \mu_{3n}$. The Galois action here is the one we considered before, and we have

$$\left(\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times^c_{\mathrm{End}(\mathcal{O}_{-m}/(3))} \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-m})}\right) \rtimes \mathbb{Z}/2\mathbb{Z} \cong ((\mathbb{Z} \times \mu_3) \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mu_n$$

(2) If $m \equiv 1 \bmod 3$, we have $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1) \cong \mathbb{F}_9$. Again the reduction map $\mu_n \cong \mathcal{O}^{\times}_{-m} \to \mathbb{F}_9^{\times}$ is injective. In any case, if c is not the orbit $\{0\}$, it consists entirely of units. Hence the fiber

product is the group of units of $\mathcal{O}_{-m}[\zeta_3]$ whose reduction is ± 1. As an abelian group, this is again isomorphic to $\mathbb{Z} \times \mu_{3n}$ with the Galois action discussed previously, hence we have

$$\left(\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-m},\zeta_3)} \times^c_{\mathrm{End}(\mathcal{O}_{-m}/(3))} \mathcal{O}^\times_{\mathbb{Q}(\sqrt{-m})}\right) \rtimes \mathbb{Z}/2\mathbb{Z} \cong ((\mathbb{Z} \times \mu_3) \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mu_n$$

Now we have all the information we need to state the computation of the Farrell–Tate cohomology of $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})})$.

Theorem 6.6. *Let $m \neq 3$ be a positive square-free integer and assume that $\mathcal{O}_{-m}$ is a principal ideal domain. Then we have*

$$\widehat{\mathrm{H}}^\bullet(\mathrm{PGL}_3(\mathcal{O}_{-m}); \mathbb{F}_3) \cong \widehat{\mathrm{H}}^\bullet(\mathbb{Z} \times \mu_3; \mathbb{F}_3)^{\oplus c \cdot h_\lambda} \oplus \widehat{\mathrm{H}}^\bullet((\mathbb{Z} \times \mu_3) \rtimes \mathbb{Z}/2\mathbb{Z}; \mathbb{F}_3)^{\oplus c \cdot h_\mu}$$

Here c is the number of $\mathcal{O}_{-m}[\zeta_3]^\times$-orbits on $\mathcal{O}_{-m}[\zeta_3]/(\zeta_3 - 1)$, h_μ is the number of ideal classes in $\mathcal{O}_{-m}[\zeta_3]$ which are Galois-invariant for the natural action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{-m}, \sqrt{-3})/\mathbb{Q}(\sqrt{-m}))$, and h_λ is the number of 2-element Galois-orbits of ideal classes. In particular, $h_{\mathbb{Q}(\sqrt{-m},\sqrt{-3})} = 2h_\lambda + h_\mu$.

PROOF OF THEOREM 1.1 AND THEOREM 6.6: Both theorems follow from Brown's formula for the Farrell–Tate cohomology. By the discussion in Section 2, the elementary abelian 3-subgroups of $\mathrm{GL}_3(\mathcal{O}_{-m})$ are all of rank 1. The structure of the centralizers and normalizers was described in the Dedekind ring case in Lemma 5.3 and Lemma 5.5. The same results are true in the more general case of Theorem 1.1, using the structure results concerning unit groups of orders, cf. [12]. Theorem 6.6 then follows from the more precise result of Proposition 4.7 and the previous discussion of the $\mathcal{O}_{-m}[\zeta_3]^\times$-action on $\mathcal{O}_{-m}/(3)$ above. ∎

More explicit information on the Farrell–Tate cohomology of such groups can now be obtained via the following computation included in [14]:

Proposition 6.7. *Let $A = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}^r$, and let ℓ be an odd prime with $\ell \mid n$. Then, with $b_1, x_1, \ldots, x_r$ denoting classes in degree 1 and a_2 a class of degree 2, we have*

$$\widehat{\mathrm{H}}^\bullet(A; \mathbb{F}_\ell) \cong \widehat{\mathrm{H}}^\bullet(\mathbb{Z}/n\mathbb{Z}; \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} \bigwedge^\bullet \mathbb{F}_\ell^r \cong \mathbb{F}_\ell[a_2, a_2^{-1}](b_1, x_1, \ldots, x_r).$$

The Hochschild–Serre spectral sequence associated to the semi-direct product $A \rtimes \mathbb{Z}/2\mathbb{Z}$ (where $\mathbb{Z}/2\mathbb{Z}$ acts as -1 on A) degenerates and yields an isomorphism

$$\widehat{\mathrm{H}}^\bullet(A \rtimes \mathbb{Z}/2\mathbb{Z}; \mathbb{F}_\ell) \cong \widehat{\mathrm{H}}^\bullet(A; \mathbb{F}_\ell)^{\mathbb{Z}/2\mathbb{Z}}.$$

The invariant classes are then given by $a_2^{\otimes 2i}$ tensor the even exterior powers plus $a_2^{\otimes(2i+1)}$ tensor the odd exterior powers.

As a direct application, the Farrell–Tate cohomology of a group like

$$\mathcal{O}_K[\zeta_\ell]^\times \rtimes \mathrm{Gal}(K(\zeta_\ell)/K) \cong (\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

with $K = \mathbb{Q}(\sqrt{-m})$ (where the action in the semidirect product on the right is consequently given by multiplication with -1) looks like the direct sum of two copies of the cohomology of the dihedral group with $2n$ elements, with one copy shifted by one.

The algebra in Theorem 6.6 is given by the $\mathbb{Z}/2\mathbb{Z}$-invariant elements in the algebra $\mathbb{F}_3[a_2^{\pm 1}](b_1, x_1)$, where the action of $\mathbb{Z}/2\mathbb{Z}$ is by multiplication with -1 on all the generators. The invariant subalgebra is then generated by the classes $b_1 x_1$ in degree 2, $b_1 a_2$ and $x_1 a_2$ in degree 3, and a_2^2 in degree 4. Consequently, the Hilbert–Poincaré series for the positive degrees is

$$2\frac{T^2 + 2T^3 + T^4}{1 - T^4} = 2\frac{T^2(1 + T)^2}{1 - T^4}.$$

6.2. **Homological 3-torsion in PGL_3 for real quadratic fields.** Now we discuss analogues of the above results for rings of integers in real quadratic fields. So let m be a positive square-free number with $3 \nmid m$, denote by $\mathcal{O}_m = \mathcal{O}_{\mathbb{Q}(\sqrt{m})}$ and assume that $\mathcal{O}_{-m}$ is a principal ideal domain. As before, conjugacy classes are parametrized by pairs $(\mathfrak{a}, c)$ with $\mathfrak{a}$ an ideal class in $\mathcal{O}_m[\zeta_3]$ and c and $\mathcal{O}_m[\zeta_3]^\times$-orbit of elements in $\mathcal{O}_m[\zeta_3]/(\zeta_3 - 1)$.

As in the imaginary case, $\mathcal{O}_m[\zeta_3]/(\zeta_3 - 1) \cong \mathcal{O}_m/(3)$ is

$$\mathbb{F}_3[X]/(X^2 - m) \cong \begin{cases} \mathbb{F}_3 \times \mathbb{F}_3 & m \equiv 1 \bmod 3 \\ \mathbb{F}_9 & m \equiv 2 \bmod 3 \end{cases}$$

An essential difference is now that the unit group $\mathcal{O}_m^\times \cong \mathbb{Z} \times \mu_2$ is already of rank 1. For the natural reduction morphism $\mathcal{O}_m^\times \to \mathcal{O}_m/(3)^\times$, various possibilities occur: for $\mathcal{O}_2$, $\mathcal{O}_5$, $\mathcal{O}_{13}$ and $\mathcal{O}_{17}$ we have

maximal possible image, while for $\mathcal{O}_7$ and $\mathcal{O}_{11}$ the image is $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$. The image of the natural reduction morphism $\mathcal{O}_m[\zeta_3]^\times \to \mathcal{O}_m/(3)^\times$ will then always contain the image of the reduction morphism $\mathcal{O}_m^\times \to \mathcal{O}_m/(3)^\times$ because $\mathcal{O}_m^\times \hookrightarrow \mathcal{O}_m[\zeta_3]^\times$ is a finite index subgroup. In particular, in those cases where already $\mathcal{O}_m^\times$ surjects onto $\mathcal{O}_m/(3)^\times$, we will have the minimal possible number of orbits.

By Dirichlet's unit theorem, we have $\mathcal{O}_m^\times \cong \mathbb{Z} \times \mu_2$ and $\mathcal{O}_m[\zeta_3]^\times \cong \mathbb{Z} \times \mu_6$. In the fiber product and up to 2-torsion, we then get (as abelian groups, but there are case distinction concerning the index in the full unit groups) $\mathbb{Z} \times \mathbb{Z} \times \mu_3$. The Galois action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{m}, \sqrt{3})/\mathbb{Q}(\sqrt{m}))$ on μ_3 is the non-trivial one mapping $\zeta_3 \mapsto \zeta_3^2$. The Galois action on $\mathbb{Z}^2$ is trivial: for the summand $\mathbb{Z} \subseteq \mathcal{O}_m^\times$ this is clear, and the summand $\mathbb{Z} \subseteq \mathcal{O}_m[\zeta_3]^\times$ contains $\mathcal{O}_m^\times$ as finite-index subgroup. Consequently, the structure of the normalizer in the real quadratic case is $\mathbb{Z}^2 \times (\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$.

For the Farrell–Tate cohomology, this implies the following result:

Proposition 6.8. *Let m be a positive square-free integer with $3 \nmid m$ and assume that $\mathcal{O}_m$ is a principal ideal domain. Then we have*

$$\widehat{\mathrm{H}}^\bullet(\mathrm{PGL}_3(\mathcal{O}_m); \mathbb{F}_3) \cong \widehat{\mathrm{H}}^\bullet(\mathbb{Z}^2 \times \mu_3; \mathbb{F}_3)^{\oplus c \cdot h_\lambda} \oplus \widehat{\mathrm{H}}^\bullet(\mathbb{Z}^2 \times (\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}); \mathbb{F}_3)^{\oplus c \cdot h_\mu}$$

Here c is the number of $\mathcal{O}_m[\zeta_3]^\times$-orbits on $\mathcal{O}_m[\zeta_3]/(\zeta_3 - 1)$, h_μ is the number of ideal classes in $\mathcal{O}_m[\zeta_3]$ which are Galois-invariant for the natural action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{m}, \sqrt{-3})/\mathbb{Q}(\sqrt{m}))$, and h_λ is the number of 2-element Galois orbits of ideal classes.

The Farrell–Tate cohomology algebra is $\mathbb{F}_3[a_2](b_1, x_1, y_1)$ for the case $\mathbb{Z}^2 \times \mu_3$ and $\mathbb{F}_3[a_2^{\pm 2}](b_1^3, x_1, y_1)$ for the case $\mathbb{Z}^2 \times (\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$.

6.3. **5-torsion in** $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{5})})$. We consider the Farrell–Tate cohomology of the group $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{5})})$. We fix the embedding $\mathbb{Q}(\zeta_5) \hookrightarrow \mathbb{C}$ given by $\zeta_5 \mapsto \exp(\frac{2\pi i}{5})$. Since $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$, 5-torsion in $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{m})})$ can only appear for the quadratic subfield $K = \mathbb{Q}(\sqrt{5})$. The minimal polynomial of ζ_5 factors as

$$\Phi_4(X) = \left(X^2 - \frac{\sqrt{5}-1}{2}X + 1\right)\left(X^2 + \frac{\sqrt{5}+1}{2}X + 1\right)$$

Since $\zeta_5 + \zeta_5^{-1} \in \mathcal{O}_{\mathbb{Q}(\sqrt{5})}$, $\zeta_5^2 = (\zeta_5 + \zeta_5^{-1})\zeta_5 - 1$ is an $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$-linear combination of 1 and ζ_5, hence $\{1, \zeta_5\}$ is a relative integral basis. We can employ Proposition 4.7 to obtain the conjugacy classification of C_5-subgroups in $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{5})})$ and to describe the Farrell–Tate cohomology.

The class group of $\mathcal{O}_{\mathbb{Q}(\zeta_5)}$ is trivial, and for an $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}[C_5]$-module M whose $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$-rank is 3 the rank of M/M_N has to be 1. Consequently, the number of conjugacy classes equals the number of $\mathcal{O}_{\mathbb{Q}(\zeta_5)}^\times$-orbits of elements in $\mathcal{O}_{\mathbb{Q}(\zeta_5)}/(\zeta_5 - 1) \cong \mathbb{F}_5$. To determine the image of $\mathcal{O}_{\mathbb{Q}(\zeta_5)}^\times \to \mathbb{F}_5^\times$ we consider the composition with the natural inclusion $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}^\times \hookrightarrow \mathcal{O}_{\mathbb{Q}(\zeta_5)}^\times$. Now we note that the fundamental unit of $\mathbb{Q}(\sqrt{5})$ is given by $\frac{\sqrt{5}+1}{2}$, whose image under the reduction map

$$\mathcal{O}_{\mathbb{Q}(\sqrt{5})} \to \mathcal{O}_{\mathbb{Q}(\sqrt{5})}/(\sqrt{5}) \cong \mathbb{F}_5$$

is $3 \in \mathbb{F}_5^\times$. Consequently, the map $\mathcal{O}_{\mathbb{Q}(\zeta_5)}^\times \to \mathbb{F}_5^\times$ is surjective and there are 2 orbits.

The structure of the normalizers and the structure of the Farrell–Tate cohomology can be determined as in the previous case of homological 3-torsion over real quadratic number rings. Consequently, the mod 5 Farrell–Tate cohomology of $\mathrm{GL}_3(\mathcal{O}_5)$ is of the form

$$\widehat{\mathrm{H}}^\bullet(\mathrm{PGL}_3(\mathcal{O}_5); \mathbb{F}_5) \cong \widehat{\mathrm{H}}^\bullet(\mathbb{Z}^2 \times (\mu_5 \rtimes \mathbb{Z}/2\mathbb{Z}); \mathbb{F}_5)^{\oplus 2}$$

6.4. **7-torsion in** $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})})$. We consider the Farrell–Tate cohomology of the group $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})})$ with $\mathbb{F}_7$-coefficients. We fix the embedding $\mathbb{Q}(\zeta_7) \hookrightarrow \mathbb{C}$, $\zeta_7 \mapsto \exp(\frac{2\pi i}{7})$. For the minimal polynomial of ζ_7 over $\mathbb{Q}(\sqrt{-7})$, we choose the first of the two factors from Example 4.1. From that polynomial, we see that ζ_7^3 is a $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$-linear combination of $1, \zeta_7$ and ζ_7^2. This implies that $1, \zeta_7, \zeta_7^2$ is a relative integral basis of $\mathbb{Z}[\zeta_7]$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$. However, in this case, the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}[T]/(\Phi_7(T))$ fails to be a Dedekind domain because the cyclotomic polynomial $\Phi_7(T)$ decomposes as product of two polynomials of degree 3 over $\mathbb{Q}(\sqrt{-7})$. We have to apply the partial result Proposition 4.7 to get a classification. This is enough for our purposes as we are interested in those modules which are $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$-free of rank 3.

The class group $\mathrm{Cl}(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})})$ is trivial, every ideal class of $\mathbb{Q}(\zeta_7)$ has a basis as $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$-module. We can therefore apply Reiner's results to determine the conjugacy classes of C_7-subgroups in $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})})$. Since the class group of $\mathrm{Cl}(\mathcal{O}_{\mathbb{Q}(\zeta_7)})$ is also trivial there is a unique conjugacy class of cyclic subgroups of order 7, corresponding to the free rank one $\mathbb{Z}[\zeta_7]$-module.

The centralizer in $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})})$ of the cyclic group of order 7 is the usual unit group

$$\mathcal{O}_{\mathbb{Q}(\zeta_7)}^\times \cong \mathbb{Z}^2 \times \mu_{14}$$

and the normalizer is an extension of this unit group by the Galois group of the extension $\mathbb{Q}(\zeta_7)/\mathbb{Q}(\sqrt{-7})$ acting in the obvious way. Therefore, the normalizer of the cyclic group is of the form

$$\mathcal{O}_{\mathbb{Q}(\zeta_7)}^\times \rtimes \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}(\sqrt{-7})) \cong \left(\mathbb{Z}^2 \times \mu_{14}\right) \rtimes \mathbb{Z}/3\mathbb{Z}.$$

The Galois action on the group μ_{14} is induced from the natural embedding $\mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathrm{Aut}(\mu_{14})$, and the action on $\mathbb{Z}^2$ is given by the 2-dimensional rotation representation.

Proposition 6.9.

$$\widehat{\mathrm{H}}^\bullet(\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}); \mathbb{F}_7) \cong \widehat{\mathrm{H}}^\bullet(\mathcal{O}_{\mathbb{Q}(\zeta_7)}^\times; \mathbb{F}_7)^{\mathbb{Z}/3\mathbb{Z}} \cong \mathbb{F}_7[a_2^{\pm 3}](b_1^5, x_1 \wedge y_1).$$

Proof. The first isomorphism follows since the Hochschild–Serre spectral sequence degenerates, essentially because the $\mathbb{F}_7$-cohomology of $\mathbb{Z}/3\mathbb{Z}$ is trivial. Now we determine the action and invariant subalgebra of

$$\begin{aligned}
\widehat{\mathrm{H}}^\bullet(\mathcal{O}_{\mathbb{Q}(\zeta_7)}^\times; \mathbb{F}_7) &\cong \widehat{\mathrm{H}}^\bullet(\mathbb{Z}^2 \times \mathbb{Z}/14\mathbb{Z}; \mathbb{F}_7) \\
&\cong \mathbb{F}_7[a_2](b_1, x_1, y_1)
\end{aligned}$$

The action on the degree 2 of $\mathrm{H}^\bullet(\mathbb{Z}/7\mathbb{Z}; \mathbb{Z}) \cong \mathbb{F}_7[a_2]$ is via the dual of the natural embedding $\mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathrm{Aut}(\mu_{14})$. The induced action in degree $2n$ of $\mathbb{F}_7[a_2^n]$ is the n-th tensor power. Therefore, the invariant subring is $\mathbb{F}_7[a_2^3]$. In the cohomology ring $\mathrm{H}^\bullet(\mathbb{Z}/7\mathbb{Z}; \mathbb{F}_7) \cong \mathbb{F}_7[a_2](b_1)$, the representation in degree $2n-1$ is the same as in degree $2n$, hence the invariant subring here is $\mathbb{F}_7[a_2^3](b_1^5)$.

The action of $\mathbb{Z}/3\mathbb{Z}$ on $\mathbb{Z}^2$ is the rotation representation. Therefore, on the cohomology $\mathrm{H}^\bullet(\mathbb{Z}^2; \mathbb{F}_7) \cong \mathbb{F}_7(x_1, y_1)$ we have the dual of the rotation representation in degree 1. The rotation is a permutation representation by $x_1 \mapsto y_1, y_1 \mapsto -x_1 - y_1$. From this, we see that $\bigwedge^2 \mathbb{Z}^2$ is the trivial $\mathbb{Z}/3\mathbb{Z}$-representation. Hence the invariant subalgebra is the exterior algebra generated by $x_1 \wedge y_1$. Combination of the two paragraphs proves the result. $\qquad\square$

The Hilbert–Poincaré series for the invariant algebra is

$$\frac{(1 + T^5)(1 + T^2)}{1 - T^6}.$$

6.5. ℓ-torsion in $\mathrm{PGL}_\ell(\mathbb{Z})$. Another example that can be handled along the lines of the $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})})$-examples is the ℓ-torsion in the case $\mathrm{PGL}_\ell(\mathbb{Z})$. By Reiner's result [15], there are $2h_{\mathbb{Q}(\zeta_\ell)}$ conjugacy classes of C_ℓ-subgroups in $\mathrm{PGL}_\ell(\mathbb{Z})$. By Dirichlet's unit theorem, $\mathcal{O}_{\mathbb{Q}(\zeta_\ell)}^\times \cong \mathbb{Z}^{\frac{\ell-3}{2}} \times \mu_{2\ell}$. Therefore, the normalizers for the two conjugacy classes belonging to a given ideal class are of the form $(\mathbb{Z}^{\frac{\ell-3}{2}} \times \mu_{2\ell}) \rtimes \mathbb{Z}/(\ell-1)\mathbb{Z}$ and $(\mathbb{Z}^{\frac{\ell-3}{2}} \times_{\mathbb{Z}/(\ell-1)\mathbb{Z}} \mu_{2\ell}) \rtimes \mathbb{Z}/(\ell-1)\mathbb{Z}$, respectively. For the special case $\ell = 5$, the Farrell–Tate cohomology of $\mathrm{PGL}_5(\mathbb{Z})$ is then twice the one for $\mathrm{PSL}_4(\mathbb{Z})$ (where the latter was computed in [6]).

7. Machine computations

The authors have run a machine computation, starting with Voronoï cell complexes with an action of $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})})$, constructed by Sebastian Schönnenbeck's software [16]. More precisely, $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})})$ acts on a certain space of quadratic forms, and there is an equivariant retraction to Ash's well-rounded retract [1]. On the latter co-compact space, a suitable form of Voronoï's algorithm yields an explicit cell structure with cell stabilizers and computable quotient space, as described by Braun, Coulangeon, Nebe and Schönnenbeck [3].

In view of determining the parameters λ and μ of Theorem 1.1, we want to extract the torsion subcomplexes from these Voronoï cell complexes.

Definition 7.1. *For ℓ a prime number, the ℓ-torsion subcomplex is the set of all cells with stabilizers containing some element(s) of order ℓ.*

For the ℓ-torsion subcomplex to be guaranteed to be a cell complex, and to consist only of fixed points of order-ℓ-elements (so to coincide with the ℓ-singular part), we need a rigidity property: We want each cell stabilizer to fix its cell pointwise. This rigidity property is lacking on our Voronoï cell complexes. In theory, it is always possible to obtain this rigidity property via the barycentric subdivision. However, the barycentric subdivision of an n-dimensional cell complex can multiply the number of cells by $(n+1)!$ and thus easily let the memory stack overflow. In previous work of the authors [6], a cell subdivision algorithm was introduced ("rigid facets subdivision"), which refines the cell structure to get the rigidity

property, in an efficient enough way to treat the $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})})$-cases. This allows us to extract the ℓ-torsion subcomplexes.

The authors applied the rigid facets subdivision algorithm to

- the $\mathrm{PGL}_3(\mathbb{Z}[i])$ cell complex of Mathieu Dutour Sikiric [7] and
- the $\mathrm{GL}_3(\mathcal{O}_{-m})$ cell complexes for $m \in \{1, 2, 5, 7, 11, 19\}$ of Sebastian Schönnenbeck [16];

then extracted the 3-torsion subcomplex, and finally reduced it using their pertinent methods [13].

Outcome 7.2. *For* $\mathrm{GL}_3(\mathcal{O}_{-m})$, $m \in \{1, 2, 5, 7, 11, 19\}$, *the 3-torsion subcomplex can be reduced to a graph. The quotient of the reduced subcomplex by the* $\mathrm{GL}_3(\mathcal{O}_{-m})$-*action consists of* λ *connected components of type* $\circ\!\!-$ *(an edge with its endpoints identified), and* μ *connected components of type* $\bullet\!\!-\!\!\bullet$ *(an edge without identifications on its endpoints). The counts of* λ *and* μ *are given in Table 1.*

Corollary 7.3. *Table 1 specifies, within its scope, the values of the parameters* λ *and* μ *in Theorem 1.1.*

Proof. On each connected component of type $\circ\!\!-$, the vertex stabilizer equals the edge stabilizer, and is of isomorphism type C_3, denoting a cyclic group with 3 elements. On each connected component of type $\bullet\!\!-\!\!\bullet$, the edge stabilizer is of isomorphism type C_3, and the two vertex stabilizers are of isomorphism type D_3, denoting a dihedral group with $3 \cdot 2$ elements, but not conjugate to each other in $\mathrm{GL}_3(\mathcal{O}_{-m})$. From previous work of the authors [6, 13], we know that since each connected component has been reduced to one edge, each conjugacy class of order-3-subgroups in $\mathrm{GL}_3(\mathcal{O}_{-m})$ is represented precisely by one of these edges' stabilizers. So we have λ, respectively μ conjugacy classes of order-3-subgroups which are not contained, respectively which are contained in a copy of D_3 in $\mathrm{GL}_3(\mathcal{O}_{-m})$. $\square$

The reduced ℓ-torsion subcomplex furthermore allows us to compute the mod ℓ Farrell-Tate cohomology of $\mathrm{GL}_3(\mathcal{O}_{-m})$ via the equivariant spectral sequence (a.k.a. the isotropy spectral sequence). Because there is up to conjugating isomorphism just one inclusion $\mathrm{C}_3 \to \mathrm{D}_3$, the $d_1^{p,q}$-differentials of the equivariant spectral sequence with $\mathbb{F}_3$-coefficients on the 3-torsion subcomplex have the maximal possible ranks, i.e., they are surjective whenever both domain of definition and codomain contain 3-torsion. The subsequent computation of the $E_2 = E_\infty$-page of the isotropy spectral sequence agrees with the results formulated in Theorem 6.6.

APPENDIX A. EXPLICIT ORDER 3 MATRICES OVER $\mathbb{Z}[\sqrt{-5}]$

We shortly discuss how arguments as in the proof of Theorem 4.4 can actually be used to produce explicit matrices of finite order. The example case we discuss is $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-5})})$, i.e., invertible 3×3-matrices of order 3 with entries in the number ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-5})} = \mathbb{Z}[\sqrt{-5}]$, which we denote by $\mathcal{O}_{-5}$ in the following. Note that our proof of Theorem 4.4 doesn't apply to that case. Nevertheless, the first part of the proof allows to provide matrices of finite order, but the conjugacy discussion in the end of the proof requires choices of suitable bases which doesn't work for non-free projective modules. In any case, we hope that the explicit arguments below clarify a bit the ideas behind the proof of Theorem 4.4 as well as the issues in the general non-PID case.

We first describe the $\mathcal{O}_{-5}$-structure of the relevant modules M over $\mathcal{O}_{-5}[\mathrm{C}_3]$. By the arguments in the proof of Theorem 4.4, the submodule $M_\mathrm{N} \subseteq M$ annihilated by the norm element has a structure of module over $\mathcal{O}_{-5}[\zeta_3]$. This is a Dedekind ring, cf. Lemma 4.2, and we have a relative integral basis $\{1, \zeta_3\}$ for $\mathcal{O}_{-5}[\zeta_3]$ over $\mathcal{O}_{-5}$. A slight change of basis allows to write

$$\mathcal{O}_{-5}[\zeta_3] \cong \mathcal{O}_{-5} \cdot \zeta_3 \oplus \mathcal{O}_{-5} \cdot (\zeta_3 - 1)$$

as $\mathcal{O}_{-5}$-modules. The ideal class group of $\mathcal{O}_{-5}[\zeta_3]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, with a representative of the nontrivial ideal class given by $(3, \sqrt{-5}\zeta_3 - 1)$. Under the above isomorphism $\mathcal{O}_{-5}[\zeta_3] \cong \mathcal{O}_{-5} \cdot \zeta_3 \oplus \mathcal{O}_{-5} \cdot (\zeta_3 - 1)$, the ideal $(3, \sqrt{-5}\zeta_3 - 1)$ maps to $(3, \sqrt{-5} - 1) \cdot \zeta_3$ in the first factor (given by setting $\zeta_3 - 1 = 0$) and to $(3, -1)$ in the second factor (given by setting $\zeta_3 = 0$). In particular, as $\mathcal{O}_{-5}$-module, we have a decomposition

$$(3, \sqrt{-5}\zeta_3 - 1) \cong (3, \sqrt{-5} - 1) \cdot \zeta_3 \oplus \mathcal{O}_{-5} \cdot (\zeta_3 - 1).$$

In particular, this implies that the relative norm map

$$\mathbb{Z}/2\mathbb{Z} \cong \mathrm{Cl}(\mathcal{O}_{-5}[\zeta_3]) \to \mathrm{Cl}(\mathcal{O}_{-5}) \cong \mathbb{Z}/2\mathbb{Z}$$

is the nontrivial map. The $\mathcal{O}_{-5}[\mathrm{C}_3]$-modules M which are free of rank 3 as $\mathcal{O}_{-5}$-modules have a direct sum decomposition $M \cong M_\mathrm{N} \oplus M/M_\mathrm{N}$ (as $\mathcal{O}_{-5}$-modules), and there are two possibilities: the first one has $M_\mathrm{N} \cong \mathcal{O}_{-5}[\zeta_3] \cong \mathcal{O}_{-5}^{\oplus 2}$ and $M/M_\mathrm{N} \cong \mathcal{O}_{-5}$, and the second one has $M_\mathrm{N} \cong (3, \sqrt{-5}\zeta_3 - 1) \cong \mathcal{O}_{-5} \oplus (3, \sqrt{-5} - 1)$ and $M_\mathrm{N} \cong (2, \sqrt{-5} + 1)$ (where we have chosen a different ideal representative $(2, \sqrt{-5} + 1)$ of the non-trivial element of the ideal class group of $\mathcal{O}_{-5}$).

The $\mathcal{O}_{-5}[C_3]$-module structures are now determined by giving the action of the element $[\gamma] \in \mathcal{O}_{-5}[C_3]$ corresponding to a chosen generator γ of C_3. By the description of the action in Theorem 4.4, the element $[\gamma]$ acts on M_N via multiplication by ζ_3. On M/M_N, the action is given by adding the image under an $\mathcal{O}_{-5}$-linear map $M/M_N \to M_N$.

In the first case, we get the following module, as well as explicit representing matrices for the order 3 elements of $\mathrm{GL}_3(\mathcal{O}_{-5})$. We have $M/M_N \cong \mathcal{O}_{-5}$ and $M_N \cong \mathcal{O}_{-5}[\zeta_3] \cong \mathcal{O}_{-5}^{\oplus 2}$. As basis of $M_N \oplus M/M_N$ we choose $1 \in M/M_N$ and $\{1, \zeta_3\}$ in M_N. The element $[\gamma] \in \mathcal{O}_{-5}[C_3]$ acts on M_N by multiplication with ζ_3, and on M/M_N by $1 \mapsto 1 + \beta$ for $\beta \in M_N$. In this basis, the representing matrix for $[\gamma]$ is

$$\begin{pmatrix} 0 & -1 & a \\ 1 & -1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The 2×2-block in the upper left is the representing matrix for multiplication by ζ_3 on $\mathcal{O}_{-5}[\zeta_3]$ with basis $\{1, \zeta_3\}$, and $a, b \in \mathcal{O}_{-5}$ determine the element $a + b\zeta_3 \in M_N \cong \mathcal{O}_{-5}[\zeta_3]$ which is the image of $1 \in M/M_N \cong \mathcal{O}_{-5}$ under the action of $[\gamma]$.

The second case is slightly more complicated. We have $M/M_N \cong (2, \sqrt{-5} + 1)$, henceforth denoted by $\mathfrak{a}$, and $M_N \cong (3, \sqrt{-5} - 1) \oplus \mathcal{O}_{-5}$, the first summand henceforth denoted by $\mathfrak{b}$. Since the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are non-principal, it is impossible to choose $\mathcal{O}_{-5}$-bases for M_N or M/M_N individually. However, we may choose a basis for $M_N \oplus M/M_N$ as follows: since the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime, we have the natural surjection

$$f \colon \mathfrak{a} \oplus \mathfrak{b} \to \mathfrak{a} + \mathfrak{b} = \mathcal{O}_{-5} \colon (x, y) \mapsto x - y.$$

The kernel of this map f is $\mathfrak{ab} = (-\sqrt{-5} + 1)\mathcal{O}_{-5}$. In particular, an $\mathcal{O}_{-5}$-basis for $\mathfrak{a} \oplus \mathfrak{b}$ is given by $(-2, -3)$ (obtained by lifting 1 along f) and $(-\sqrt{-5} + 1, -\sqrt{-5} + 1)$ (obtained as image of the generator of $\ker f$ under the inclusion into $\mathfrak{a} \oplus \mathfrak{b}$). We can now write down a choice of basis for

$$M_N \oplus M/M_N \cong \mathfrak{b} \cdot \zeta_3 \oplus \mathcal{O}_{-5} \cdot (\zeta_3 - 1) \oplus \mathfrak{a}$$

given by $(-3 \cdot \zeta_3, 0, -2)$, $(0, 1, 0)$ and $((-\sqrt{-5} + 1) \cdot \zeta_3, 0, -\sqrt{-5} + 1)$. Note that the decomposition $M_N \cong \mathfrak{b} \cdot \zeta_3 \oplus \mathcal{O}_{-5} \cdot (\zeta_3 - 1)$ is the one induced from the decomposition $\mathcal{O}_{-5}[\zeta_3] \cong \mathcal{O}_{-5} \cdot \zeta_3 \oplus \mathcal{O}_{-5} \cdot (\zeta_3 - 1)$ corresponding to the relative integral basis $(\zeta_3, \zeta_3 - 1)$.

The action of $[\gamma] \in \mathcal{O}_{-5}[\zeta_3]$ is now given as follows: on M_N, $[\gamma]$ acts by multiplication with ζ_3; and on $M/M_N \cong \mathfrak{a}$, $[\gamma]$ acts by sending an element $x \in \mathfrak{a}$ to $x + \lambda(x)$ where $\lambda \colon \mathfrak{a} \to M_N \cong \mathfrak{b} \cdot \zeta_3 \oplus \mathcal{O}_{-5} \cdot (\zeta_3 - 1)$ is an $\mathcal{O}_{-5}$-linear map. To describe such a linear map, we note that we have an isomorphism

$$\phi \colon \mathfrak{a} \to \mathfrak{b} \colon 2 \mapsto -(\sqrt{-5} - 1), \sqrt{-5} + 1 \mapsto 3.$$

The fact that this is a well-defined map (as well as the well-definedness of the inverse) follows from the well-known equality $2 \cdot 3 + (\sqrt{-5} + 1)(\sqrt{-5} - 1) = 0$. In particular, the composition $\mathfrak{a} \xrightarrow{\lambda} M_N \xrightarrow{\mathrm{pr}_1} \mathfrak{b}$ is an $\mathcal{O}_{-5}$-multiple of this isomorphism (because $\mathrm{Hom}_{\mathcal{O}_{-5}}(\mathfrak{a}, \mathfrak{b}) \cong \mathfrak{a}^\vee \otimes \mathfrak{b} \cong \mathcal{O}_{-5}$). The second component of a linear map $\lambda \colon \mathfrak{a} \to M_N$ is a linear form $\mathfrak{a} \to \mathcal{O}_{-5}$ which can be described as follows: from before, we know $\mathfrak{a} \otimes \mathfrak{b} = \mathfrak{ab} = (-\sqrt{-5} + 1)\mathcal{O}_{-5}$, hence a linear map $\mathfrak{a} \to \mathcal{O}_{-5}$ always has the form

$$\psi \colon \mathfrak{a} \to \mathcal{O}_{-5} \colon x \mapsto \frac{x \cdot y}{(-\sqrt{-5} + 1)}$$

for an element $y \in \mathfrak{b}$. More precisely, for an element $3a + (\sqrt{-5} + 1)b$ with $a, b \in \mathcal{O}_{-5}$, we have $\psi(2) = (-1 - \sqrt{-5})a - 2b$ and $\psi(\sqrt{-5} + 1) = (2 - \sqrt{-5})a + (1 + \sqrt{-5})b$.

We are now ready to combine everything to obtain representing matrices representing multiplication with $[\gamma]$ in the respective $\mathcal{O}_{-5}[C_3]$-module structures. We use the basis $(-3 \cdot \zeta_3, 0, -2)$, $(0, 1, 0)$ and $((-\sqrt{-5} + 1) \cdot \zeta_3, 0, -\sqrt{-5} + 1)$ given above. Multiplication by $[\gamma]$ is given by multiplication with ζ_3 on the first two summands and by addition of $\lambda(x)$ on the third summand. This leads to the following

$$\begin{pmatrix} (1 - \sqrt{-5})x - 8 & 3 & (\sqrt{-5} + 2)x - 3\sqrt{-5} + 3 \\ -3 - (1 + \sqrt{-5})a + 2b & 1 & -\sqrt{-5} + 1 + 3a + (\sqrt{-5} - 1)b \\ 2x - 3\sqrt{-5} - 3 & \sqrt{-5} + 1 & (\sqrt{-5} - 1)x + 7 \end{pmatrix}$$

where $x, a, b \in \mathcal{O}_{-5}$ give rise to the linear map

$$\lambda \colon \mathfrak{a} \to \mathfrak{b} \oplus \mathcal{O}_{-5} \colon z \mapsto \left(x \cdot \phi(z), \frac{z(3a + (\sqrt{-5} + 1)b)}{(-\sqrt{-5} + 1)} \right).$$

According to Theorem 4.4 resp. Proposition 4.7, the matrices written above provide representatives for all conjugacy classes of C_3-subgroups in $\mathrm{GL}_3(\mathcal{O}_{-5})$.

It remains to discuss conjugacy relations among the infinitely many matrices written out above. The first case corresponding to the principal ideal class is easier: conjugation by $e_{13}(b)$ and $e_{23}(x)$ allows to reduce to matrices of the form

$$\begin{pmatrix} 0 & -1 & a \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the conjugacy class only depends on the residue of $a \in \mathcal{O}_{-5}$ modulo (3). Moreover, as in the proof of Theorem 4.4, scaling by global units from $\mathcal{O}_{-5}[\zeta_3]$ is allowed. In particular, we get 5 conjugacy classes, using the representatives $a \in \{0, 1, \sqrt{-5}, 1+\sqrt{-5}, 1-\sqrt{-5}\}$ of the $\mathcal{O}_{-5}[\zeta_3]^\times$-orbits on $\mathcal{O}_{-5}/(3)$. We can also see that conjugation by

$$\begin{pmatrix} 0 & 1 & a \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

takes the square of a matrix as above (with $(3,1)$-entry a) to one with $(3,1)$-entry $2a$ which mod 3 is in the same global unit orbit as the original matrix. In particular, all the above matrices are conjugate to their squares, hence the corresponding cyclic groups embed in dihedral groups.

The conjugacy relations between the matrices arising from the non-trivial ideal class are not so easy to identify. In particular, choosing suitable bases as before is significantly more difficult because we can only choose a basis in the full module $M_N \oplus M/M_N$ but not in the individual summands. A discussion as before for the trivial ideal class is therefore not possible. At this point, we do not know what the appropriate generalization of Theorem 4.4 or Proposition 4.7 should be, and how to properly identify the number of conjugacy classes of elements resp. subgroups. The computer calculations for the case $\mathcal{O}_{-5}$ suggest that there are six conjugacy classes of order 3 elements related to the non-trivial ideal class; and the Galois group of $\mathbb{Q}(\zeta_3, \sqrt{-5})/\mathbb{Q}(\sqrt{-5})$ fixes four of these classes (which hence acquire a dihedral overgroup) and interchanges the remaining two.

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