Non-semisimple TQFT & manifold invariants

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joint work with

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Outline: for finite ribbon cat. + extra propropenties:

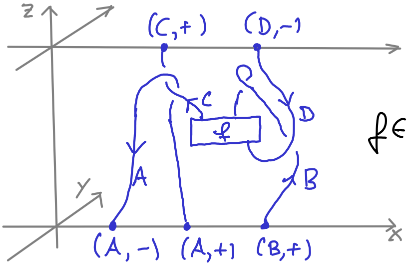
invariants of ribbon graphs from modified trace

$\rightsquigarrow$ invariants of closed 3mf + ribbon graphs $\rightsquigarrow$ 3d TQFT

Reshetikhin – Turaev functor

$\mathcal{R}$: ribbon cat.

$\mathrm{Rib}_{\mathcal{R}}$: $\mathcal{R}$-coloured ribbon tangles in $\mathbb{R}^2 \times [0,1]$ / isotopy



$f \in \mathcal{C}(D^*, C B^*)$

Get a canonical ribbon functor $F : \mathrm{Rib}_{\mathcal{R}} \to \mathcal{R}$

"make it a string diagram"

If $\mathrm{End}_{\mathcal{R}}(\mathbb{1}) = \mathbb{C}$:

closed tangle $\Gamma \longmapsto$ number $F(\Gamma)$

Example: symplectic fermions

Davydov, IR '12, IR '12

$N = 1, 2, \dots$

$\beta^4 = (-1)^N$

$SF_{N,\beta}$ **modular tensor cat.**

($\mathbb{C}$-lin. ribbon tensor cat. with non-degenerate braiding)

$\cong$ $\mathbb{C}$-lin.

$$\underbrace{\mathrm{Rep}_{\mathrm{SVec}}\, Gr_{2N}}_{SF_0} \oplus \underbrace{\mathrm{SVec}}_{SF_1}$$

4 simples:

$\mathbb{1} = \mathbb{C}^{1|0}$ $\qquad$ $T = \mathbb{C}^{1|0}$

$\Pi\mathbb{1} = \mathbb{C}^{0|1}$ $\qquad$ $\Pi T = \mathbb{C}^{0|1}$

projective

E.g. $T \otimes T = Gr_{2N}$

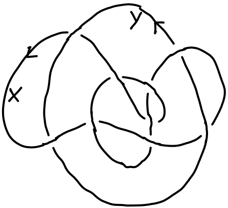
... symplectic fermions

Gainutdinov, Farsad, IR '17

- Can give factorisable ribbon quasi-Hopf alg. $Q_{N,\beta}$ s.th. $SF_{N,\beta} \simeq \mathrm{Rep}\, Q_{N,\beta}$ as ribbon cat

obj	$\mathbb{1}$	$\Pi\mathbb{1}$	T	ΠT
twist Θ	1	1	β^{-1}	$-\beta^{-1}$
dim	1	-1	0	0

Boring invariants



$\Gamma \in \mathrm{Rib}_{SF_{N|\beta}}$

assume Γ is a link:
- closed
- no coupons

Berger, Gainutdinov, IR, in prep.

- Γ contains colour T, ΠT: $F(\Gamma) = 0$
- Γ only contains col. in $SF_0 = \mathrm{Rep}_{sVec}\, Gr_{2N}$

$\hookrightarrow$ factors through $Gr(SF_0) \simeq \mathbb{C}[\mathbb{Z}/2]$

$\hookrightarrow$ same inv. as for SVec

Modified trace

Geer, Kujawa, Patureau-Mirand '11

$\mathcal{C}$: finite ribbon cat.

$\mathcal{J} \subset \mathcal{C}$: tensor ideal

- full subcat.
- $\otimes : \mathcal{J} \times \mathcal{C} \longrightarrow \mathcal{J}$
- closed under direct summands

- Examples: $\mathcal{J} = \mathcal{C}$, $\mathcal{J} = \operatorname{Proj} \mathcal{C}$ projective ideal

- Lemma: For all ideals $\mathcal{J} \neq 0$: $\operatorname{Proj} \mathcal{C} \subset \mathcal{J}$

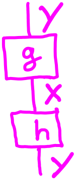
- For $\mathcal{C}$ s.si : $\operatorname{Proj} \mathcal{C} = \mathcal{C}$ $\rightsquigarrow$ no interesting ideals

... modified trace

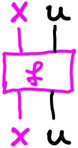
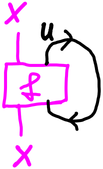
Modified trace on $\mathcal{J}$ is:

$$\{ t_X : \mathrm{End}(X) \longrightarrow k \mid X \in \mathcal{J} \}$$

1) cyclic

$t_X($  $)$ $=$ $t_Y($  $)$ $\quad X, Y \in \mathcal{J}$

2) partial trace

$t_{X \otimes U}($  $)$ $=$ $t_X($  $)$ $\quad X \in \mathcal{J}$, $U \in \mathcal{C}$

. . . modified trace

$\mathcal{J} = \mathcal{C}$: mod. tr $\propto$ categorical trace

Berger, Gainutdinov, IR, in prep.

<u>Lem.</u> $I \subsetneq \mathcal{C}$ ideal ($\Rightarrow \mathcal{C}$ non-ssi), $X \in I$, $f \in \mathrm{End}(X)$.
Categorical trace is zero : $\mathrm{tr}_X f = 0$.

$\mathcal{J} = \mathrm{Proj}\,\mathcal{C}$:

$$\boxed{\begin{array}{ll} \mathbb{1} \hookrightarrow P_{\mathbb{1}} & \text{proj. cover of } \mathbb{1} \\ \qquad\quad \downarrow & \text{has socle } \mathbb{1} \\ \qquad\quad \mathbb{1} & \end{array}}$$

Geer, Kujawa, Patureau-Mirand '18

<u>Thm</u> $\mathcal{C}$: finite ribbon cat., unimodular.
There exists a up to scalars unique mod. tr. on $\mathrm{Proj}\,\mathcal{C}$.
This mod. tr. gives non-deg. pairings on Hom's.

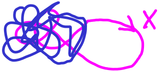
Ribbon graph invariants from modified traces

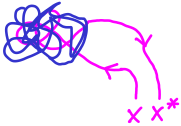
Geer, Patureau-Mirand, Virelizier '11

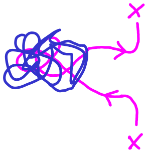
$\mathcal{C}$: finite ribbon cat., $\mathcal{J} \subset \mathcal{C}$: ideal, t : mod. tr. on $\mathcal{J}$

Closed graph Γ admissible if at least one edge coloured in $\mathcal{J}$.

Let $X \in \mathcal{J}$

$\Gamma =$ 

$\longrightarrow$ cutting presentation $\Gamma_X =$ 

Renormalised invariant $F'(\Gamma) := t_X\Big($ 

$\Big)$

Thm.: $F'(\Gamma)$ is an isotopy invariant.

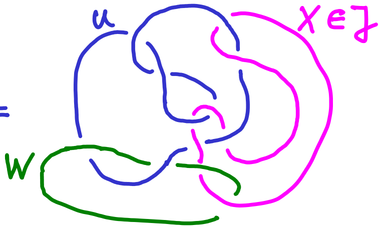
Some properties of renorm. invariants

Berger, Gainutdinov, IR, in prep.

- If W-col. loop is not cut in $F'(\Lambda(w)) = tr_X \Lambda(w)_X$ then

  $[w] = [w']$ in $Gr\,\mathcal{C}$

  $\Rightarrow \quad F'(\Lambda(w)) = F'(\Lambda(w'))$

$\Lambda(W) =$ 

- Let $W(\lambda)$, $\lambda \in \mathbb{C}$ be a family of mutually non-iso. obj. with $[W(\lambda)] \in Gr\,\mathcal{C}$ indep. of λ. Then ren. inv. for

  $\mathcal{J} = \mathcal{C}$ and $\mathcal{J} = Proj\,\mathcal{C}$

  cannot detect λ.

Example: Symplectic fermions $SF_{N,\beta}$

Berger, Gainutdinov, IR, in prep.

Take $N=2$ and consider 3 ideals:

$$\underbrace{\mathcal{J}_0 = SF_{2,\beta} \supsetneq \mathcal{J}_1}\supsetneq \mathcal{J}_2 = \mathrm{Proj}\, SF_{2,\beta}$$

- $SF_{N,\beta} = \mathrm{Rep}\, Q_{N,\beta}$ ribbon quasi-Hopf algebra in Vect generated by $f_1^\pm, f_2^\pm, K$
- $A \hookrightarrow Q_{2,\beta}$ sub quasi-Hopf alg. gen. by $f_1^\pm, K$
- $R : \mathrm{Rep}\, Q_{2,\beta} \xrightarrow{\otimes} \mathrm{Rep}\, A$ $\quad \mathcal{J}_1 = R^*(\mathrm{Proj}\, A)$ full subcat. of obj. s.th. ...

Fontalvo Orozco, Gainutdinov '18

- Pull back mod. tr. from Proj A to $\mathcal{J}_1$

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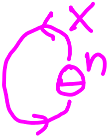
$\Gamma =$ 

$\mathcal{J}_0 \longrightarrow$

X	$\mathbb{1}$	$\Pi\mathbb{1}$	T	ΠT
$F(\Gamma)$	1	-1	0	0

$\mathcal{J}_1$

$\mathcal{J}_2$

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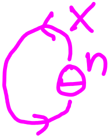
$\Gamma =$ 

$\mathcal{J}_0$

$\mathcal{J}_1$

$\mathcal{J}_2$

X	$P_{\mathbb{1}}$	$P_{\pi\mathbb{1}}$	T	πT
$F(\Gamma)$	$2n^2\beta^2$	$\dots$	$\frac{1}{2}\beta^n$	$\dots$

Example: Symplectic fermions $SF_{N,\beta}$

Berger, Gainutdinov, IR, in prep.

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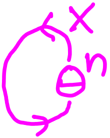
$\Gamma =$ 

$\mathcal{J}_0$

$\mathcal{J}_1$

X	$Q_M \qquad M \in \mathrm{Mat}(2,\mathbb{C})$
$F(\Gamma)$	$-\frac{n}{2}\left(1+\det M\right)\left(1+i\beta^2\right)$

$\mathcal{J}_2$

Q: What can intermediate ideals detect?

Coends and bichrome graphs

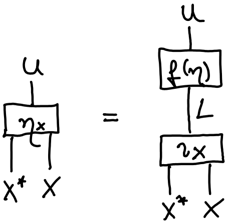
Majid '93
Lyubashenko '94

$\mathcal{C}$: unimodular finite ribbon cat.

• coend $L = \int^{X\in\mathcal{C}} X^*\otimes X$

& $i_X : X^*\otimes X \to L$

For every dinatural family $\eta_X : X^*\otimes X \to U$



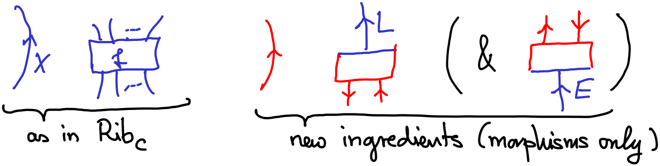
• L is Hopf alg. in $\mathcal{C}$. Pick non-zero integral $\Lambda : \mathbf{1} \to L$.

(exists & unique up to scalar)

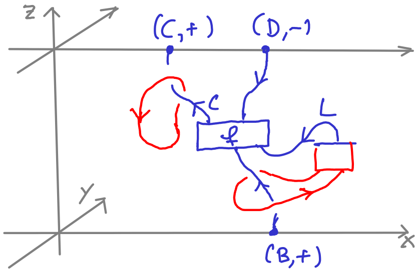
If $\mathcal{C}$ is semisimple: $L = \bigoplus_{i\in \mathrm{Irr}\,\mathcal{C}} i^*\otimes i$, $\Lambda = \sum_{i\in \mathrm{Irr}\,\mathcal{C}} \dim i \cdot$ 

... coends & bichrome graphs

Rib_Λ : ribbon cat. of bichrome graphs



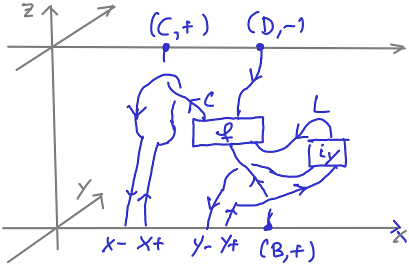
E.g.:

$T =$ 

... coends & bichrome graphs

Extend functor F to $F_\Lambda : \mathrm{Rib}_\Lambda \to \mathcal{C}$: To get $F_\Lambda(T)$,

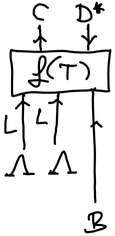
1) cut red ribbons, get dinat. xfer

$F\Big($ 

$\Big) : \underbrace{X^*X\,Y^*Y}_{\text{dinatural}}B \to CD^*$

2) $\exists!\ f(T) : LLB \longrightarrow CD^*$

3) Compose with Λ

$F_\Lambda(T) :=$ 

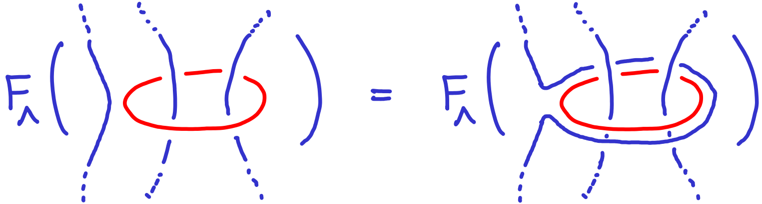
... coends & bichrome graphs

Hennings '96, Lyubashenko '94
De Renzi, Geer, Patureau-Mirand '17
De Renzi, Geer, Patureau-Mirand, Gainutdinov, IR '19

Thm.: $\mathcal{C}$: unimodular finite ribbon cat.

1) $F_\Lambda : \mathrm{Rib}_\Lambda \longrightarrow \mathcal{C}$ is an isotopy inv.
2) F_Λ is invariant under handle slides of red / blue components along a red loop

E.g.:



Q: Apart from (L, Λ), what other solutions to
are there?

3-manifold invariants

De Renzi, Geer, Patureau-Mirand '17
De Renzi, Geer, Patureau-Mirand, Gainutdinov, IR '19

$\mathcal{C}$: unimodular finite ribbon cat.

$\mathcal{J} \subset \mathcal{C}$: ideal $\qquad$ t : mod. tr. on $\mathcal{J}$

Require $\mathcal{C}$ to be twist-nondegenerate : $\Delta_{\pm} := F_{\Lambda}(\bigcirc^{\pm 1}) \neq 0$

Let (M, Γ) be a closed 3mf with embedded ribbon graph admissible w.r.t. $\mathcal{J}$

$$L'(M,\Gamma) := \mathcal{D}^{-1-|L|}\, \delta^{\sigma(L)}\, F'_{\Lambda}(L \cup \Gamma)$$

$\mathcal{D}^2 = \Delta^+ \Delta^-$ $\qquad$ $\delta = \Delta^+/\mathcal{D}$ $\qquad$ surgery link for M

... 3mf invariants

Thm.: $L'(M,\Gamma)$ is a topological invariant.

Examples: Assume $\mathcal{C}$ non-ssi. $P \in \mathcal{C}$ projective

$\mathcal{J} = \mathcal{C}$: $\to$ Lyubashenko '94

$$L'(S^2 \times S^1) = 0$$

$\mathcal{J} = \mathrm{Proj}\,\mathcal{C}$:

$$L'(\underbrace{S^2 \times S^1}_{\text{⚠ not adm.}})$$

... 3 mf invariants

Thm.: $L'(M,\Gamma)$ is a topological invariant.

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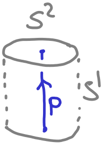
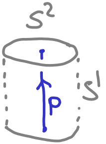
$L'\Big($ 

$\Big) = 0$

$\mathcal{J} = \mathrm{Proj}\, \mathcal{C}$:

$$L'(\underbrace{S^2 \times S^1}_{\text{⚠ not adm.}})$$

$L'\Big($ 

$\Big)$

$= \dim \mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1})$

... 3 mfd invariants

- $L'_{\mathcal{J}=\mathrm{Proj}\,\mathcal{C}}(M,\Gamma) = L'_{\mathcal{J}=\mathcal{C}}(M)\cdot L'_{\mathcal{J}=\mathrm{Proj}\,\mathcal{C}}(S^3,\Gamma)$

connected

adm. graph contained in 3-ball

- E.g. $SF_{N,\beta}$ $\quad L'_{\mathcal{J}=\mathcal{C}}(\text{Lens space}_{1,p}) = \beta^2 (2p)^N$

⚠ For $\mathcal{C}$ ssi, L' can never distinguish all $(1,p)$ lens spaces

Non-ssi TQFT

$\mathcal{C}$: modular tensor cat. $\qquad \mathcal{J} = \mathrm{Proj}\,\mathcal{C}$

central extension to cancel anomaly

$$\widehat{\mathrm{Bord}}_3(\mathcal{C})$$

3d bordisms with embedded $\mathcal{C}$-coloured ribbon graphs

$$\cup$$

$$\widehat{\mathrm{Bord}}_3'(\mathcal{C})$$

restrict to admissible bordisms

- same objects
- bordisms must have a projective ribbon in every connected component disjoint from incoming boundary

... non-ssi TQFT

E.g

- $\emptyset \xrightarrow{M} \emptyset$ must contain ribbon col. by $P \in \mathrm{Proj}\,\mathcal{C}$

- 

  $\Sigma \sqcup -\Sigma \longrightarrow \emptyset$ ✓

  $\emptyset \longrightarrow \Sigma \sqcup -\Sigma$ ✗

but can have 

for P projective.

… non-ssi TQFT

De Renzi, Geer, Patureau-Mirand '17
De Renzi, Geer, Patureau-Mirand, Gainutdinov, IR '19

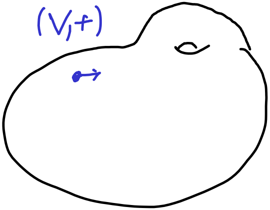
Thm.: From modular tensor cat. $\mathcal{C}$ get sym. mon. functor

$$Z : \widehat{\mathrm{Bord}}_3'(\mathcal{C}) \longrightarrow \mathrm{vect}$$

Obtained from universal construction

"closed invariants $\rightsquigarrow$ functor"

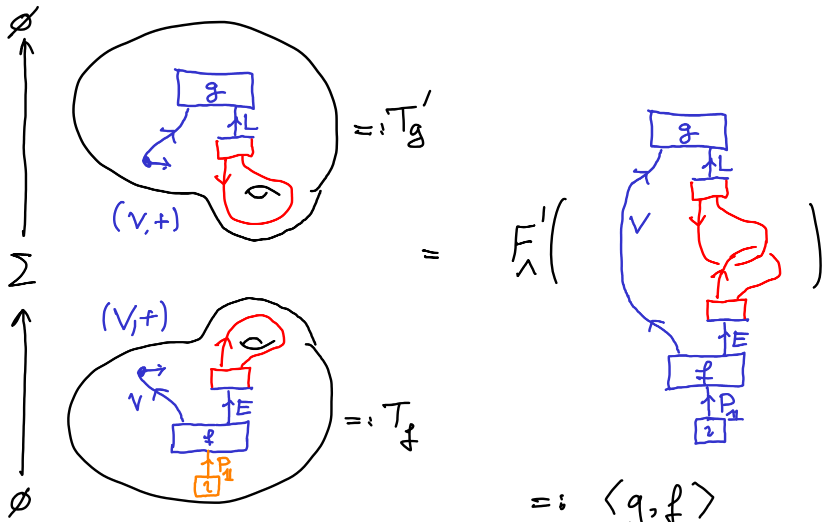
- state space



$\longmapsto \mathcal{C}(P_1, V\,E)/\sim$

... non-ssi TQFT

- Pairing with $\mathcal{C}(\vee L, \mathbb{1})$ via Heegard splitting of S^3



$=: \langle g, f \rangle$

... non-ssi TQFT

- For $\Sigma \xrightarrow{M} \Sigma$ define $Z(M)$ via

$$\langle g, Z(M) f \rangle := \delta^{n} F'_{\Lambda}\left(T'_{g} \circ M \circ T_{f}\right)$$

from $\widehat{\mathrm{Bord}}_3$

Summary

- $\mathcal{C}$: unimodular finite ribbon

  $F_\Lambda : \mathrm{Rib}_\Lambda \longrightarrow \mathcal{C}$

- $\mathcal{C}$: ... & twist-nondeg. & $\mathcal{J} \subset \mathcal{C}$, mod. tr. t

  L' inv. of 3mf with admissible rib. graph

- $\mathcal{C}$: modular tensor cat., $\mathcal{J} = \mathrm{Proj}\,\mathcal{C}$

  $Z : \widehat{\mathrm{Bord}}'_3 \longrightarrow \mathrm{vect}$