Invariants of 4-manifolds from Khovanov-Rozansky link homology

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Motivation

How does Khovanov homology extend to other ambient manifolds?

Hints:

- Functoriality under link cobordisms in 4d.
- Rozansky&Willis invariants for nullhomologous links in $\#^k(S^1 \times S^2)$.
- Rasmussen: Kh sensitive to smooth surfaces in B^4.

Proposal: $\mathrm{Kh}(L) = \mathrm{Kh}(B^4; L)$

- 4-manifolds (with (link in) boundary) $\to$ chain complexes
- 3-manifolds $\to$ dg categories

  $\vdots$
- point $\to$ some 4-category

Today: a few steps in this direction.

Starting in dimension 3...

Link invariants

The $\mathfrak{gl}_N$ link polynomial $P_N\colon \{\text{framed, oriented links}\} \to \mathbb{Z}[q^{\pm 1}]$:

$$P_N(\overcrossing) - P_N(\undercrossing) = (q - q^{-1}) P_N(\smoothing)$$

$$P_N(\kink) = q^N P_N(\strand), \quad P_N(L_1 \sqcup L_2) = P_N(L_1) P_N(L_2)$$

Higher categories

Ribbon category $\mathrm{Rep}(U_q(\mathfrak{gl}_N))$, tangle invariants

Manifold invariants

The $\mathfrak{gl}_N$ skein module for compact, oriented M^3, $P \subset \partial M^3$:

$$\mathrm{Sk}_N(M^3; P) := \frac{\mathbb{Z}[q^{\pm 1}]\langle \text{framed, oriented tangles in } (M^3, P) \rangle}{\langle \text{isotopy, local relations in } B^3 \hookrightarrow M^3 \rangle}$$

Part of a 0123ε-dimensional TFT.

...upgrading to dimension 4

Khovanov–Rozansky 2004, Robert–Wagner+Ehrig–Tubbenhauer–W 2017:

Link invariants

The $\mathfrak{gl}_N$ Khovanov–Rozansky link homology

$$\mathrm{KhR}_N\colon \{\text{links/link cobordisms}\} \to K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect}), \quad \chi_q \circ \mathrm{KhR}_N = P_N$$

Morrison–Walker–W 2019:

Higher categories

A ribbon 2-category / a disk-like 4-category categorifying $\mathrm{Rep}(U_q(\mathfrak{gl}_N))$.

Manifold invariants

A 'skein module' $\mathcal{S}_N(W^4; L)$ for compact, oriented, smooth W^4, $L \subset \partial W^4$.

$$\mathcal{S}_N(B^4; L) \cong \mathrm{KhR}_N(L).$$

Part of a 01234ε-dimensional TFT?

Approaches

Some routes to Khovanov–Rozansky homology for (links in) 3-manifolds:

- Categorify Witten–Reshetikhin–Turaev invariants
  - Categorification at roots of unity
  - Categorification of tensor product reps
- Categorify skein modules
  - Via surgery
  - Via Heegaard splitting, categorified skein algebras
- Extending Witten's model for Khovanov homology in $\mathbb{R}^3$
- Higher skein modules (this talk)
  - Functorial tangle invariant $\to$ 4-category $\to$ skein module

Khovanov–Rozansky homology

$$\left\{\begin{matrix}\text{links diagrams} \\ \text{movies of diagrams/m. moves}\end{matrix}\right\} \xrightarrow{\mathrm{KhR}_N} K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect})$$

$$\updownarrow \cong \qquad\qquad \downarrow \chi_q$$

$$\left\{\begin{matrix}\text{links embedded in } B^3 \\ \text{cobordisms in } B^3 \times I\text{/isotopy}\end{matrix}\right\} \xrightarrow{P_N \circ K_0} \mathbb{Z}[q^{\pm 1}]$$

Defining KhR_N requires:

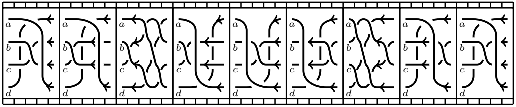
- the data of a chain complex for each link diagram (KhR04, RW17)
- the data of a chain map for every elementary movie (KhR04)
- movie move checks (Blanchet10, ETW17)

$\Longrightarrow$ KhR_N can be considered as diagram-independent (MWW19).

Khovanov–Rozansky homology

$$\left\{ \begin{array}{c} \text{links diagrams} \\ \text{movies of diagrams/m. moves} \end{array} \right\} \xrightarrow{\mathrm{KhR}_N} K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect})$$

E.g. this chain map should be homotopic to the identity:



MM10

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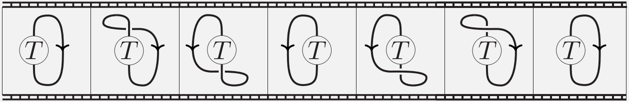
Functoriality in S^3

For $\mathcal{S}_N(B^4; L) \cong \mathrm{KhR}_N(L)$ we need KhR_N for links in $S^3 = B^3 \cup \{\infty\}$.

- links in S^3 generically avoid ∞
  $\implies$ same chain complexes
- link cobordisms in $S^3 \times I$ generically avoid $\infty \times I$
  $\implies$ same chain maps
- link cobordism isotopies in $S^3 \times I^2$ might intersect $\infty \times I^2$ transversely
  $\implies$ a new movie move to check, non-local if viewed from B^3

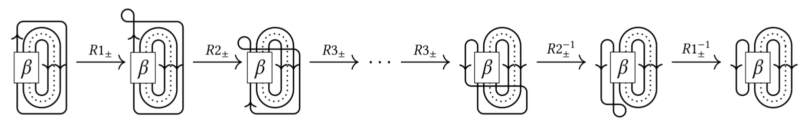


Theorem (M.-W.-W. 2019)

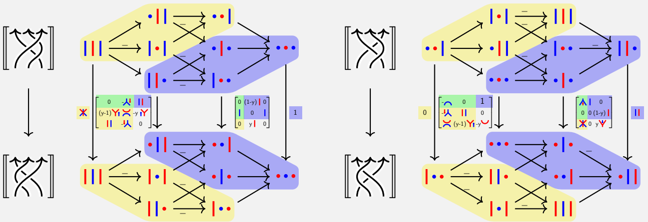
KhR_N is invariant under the sweeparound move, thus functorial in S^3.

Proving the sweeparound move

1. Reduce to the case of almost braid closures
2. Compare front and back versions of



3. Consider filtration by homological degree of extra crossings
4. Front and back versions of R1, R2, R3 agree* in associated graded



Ribbon 2-category via KhR_N for tangles

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{tangle diagrams}\\ \text{movies of diagrams/m. moves}\end{array}\right\} & \xrightarrow{[\![-]\!]_N} & K^b(N\mathbf{Foam}) \\
\updownarrow \cong & & \downarrow \chi_q \\
\left\{\begin{array}{c}\text{tangles embedded in } B^3\\ \text{cobordisms in } B^3\times I\text{/isotopy}\end{array}\right\} & \xrightarrow{RT_N \circ K_0} & \mathrm{Rep}(U_q(\mathfrak{gl}_N))
\end{array}$$

Theorem (M.-W.-W. 2019)

$\exists$ linear braided monoidal 2-category (Kapranov–Voevodsky, Baez–Neuchl, Day–Street, Crans) with duals (Barrett–Meusburger–Schaumann) with

- Objects: tangle boundary sequences
- 1-morphisms: Morse data for tangle diagrams
- 2-morphisms from T_1 to T_2: $\mathrm{H}^*\mathrm{Ch}(N\mathbf{Foam})([\![T_1]\!]_N, [\![T_2]\!]_N)$.

Towards TFT

Questions

Is this braided monoidal 2-category (or something similar) 4-dualizable and $SO(4)$-fixed in a suitable 5-category of E_2 2-categories? What is the role of the sweeparound move? Can this all be made homotopy-coherent?

$\implies$ a local 01234ε-d oriented TFT via the cobordism hypothesis.

Proposed direct construction for the 4ε part (on the level of homology):

Theorem (M.-W.-W. 2019)

KhR_N controls a disk-like 4-category, determines $\mathcal{S}_N(W^4; L)$ via the blob complex (Morrison–Walker 2010).

Rest of the talk: focus on degree zero blob homology $\mathcal{S}_N^0(W^4; L)$.

A skein module for 4-manifolds

In analogy to

$$\mathrm{Sk}_N(M^3;P) := \frac{\mathbb{Z}[q^{\pm 1}]\langle \text{framed, oriented tangles in } (M^3,P)\rangle}{\langle \ker RT_N \text{ in } B^3 \hookrightarrow M^3\rangle}$$

we would like to define $\mathcal{S}_N^0(W^4;L)$ as:

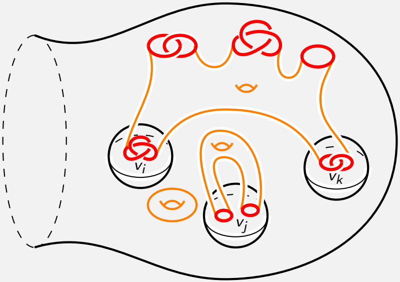
$$\frac{\mathbb{Z}\langle \text{framed, oriented surfaces in } (W^4,L)\rangle}{\langle \ker [\![-]\!]_N \text{ in } B^4 \hookrightarrow W^4\rangle}$$

Problem: Want $\mathcal{S}_N(B^4;L) \cong \mathcal{S}_N^0(B^4;L) \cong \mathrm{KhR}_N(L)$, but this is not always spanned by images of cobordisms maps.

$\implies$ consider decorated framed, oriented surfaces.

Skeins

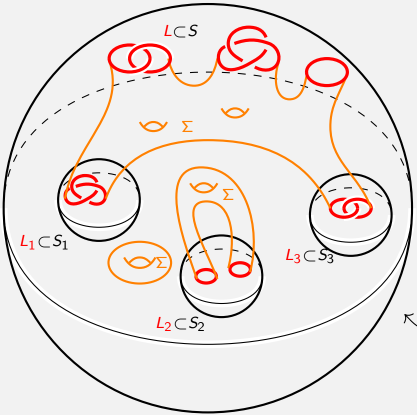
A lasagna filling of W^4 with a link $L \subset \partial W^4$ is the data of:



B_i^4 : finitely many disjoint 4-balls in ${W^4}^\circ$
L_i : input links in ∂B_i^4
Σ : f., o. surface in $(W^4 \setminus \sqcup_i B_i^4; L \sqcup_i L_i)$
$v_i \in \mathrm{KhR}_N(\partial B_i^4, L_i)$

Skein relations via lasagna algebra

Khovanov–Rozansky homology is an algebra for the lasagna operad



$$\to \begin{pmatrix} \mathrm{KhR}_N(S^3, L) \\ \uparrow \mathrm{KhR}(D) \\ \bigotimes_i \mathrm{KhR}_N(S_i, L_i) \end{pmatrix}$$

↖ A lasagna diagram D with
L_i, L : input/output links
Σ : f., o. surface in $(B^4 \setminus \sqcup_i B^4_i; L \sqcup_i L_i)$

Note: A lasagna filling of (B^4, L) is a lasagna diagram D plus $(v_1, \ldots, v_r)$.
$\implies$ evaluates to $\mathrm{KhR}_N(D)(v_1 \otimes \cdots \otimes v_r) \in \mathrm{KhR}(\partial B^4, L)$.

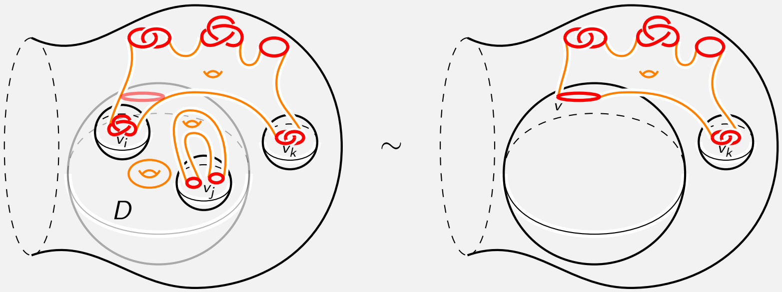
Definition of $\mathcal{S}_N^0(W^4; L)$

Definition

We define the $H_2(W^4, L) \times \mathbb{Z}_q \times \mathbb{Z}_t$-graded abelian group

$$\mathcal{S}_N^0(W^4; L) := \mathbb{Z}\langle \text{lasagna fillings of } (W^4, L)\rangle / \sim$$

where the 'skein relations' $\sim$ are generated by



with $v = \mathrm{KhR}(D)(v_i \otimes \cdots \otimes v_j)$.

To finish, some examples

Example (B^4)

$\mathcal{S}_N(B^4; L) \cong \mathcal{S}_N^0(B^4; L) \cong \mathrm{KhR}(L)$ by construction.

Example ($B^3 \times S^1$)

$\mathcal{S}_2(B^3 \times S^1; L)$ is related to the Hochschild homology of Khovanov's arc algebra and to Rozansky's homology theory for links L in $S^2 \times S^1$.

Theorem (Manolescu–Neithalath 2020)

If W^4 is a 2-handle body with a single 0-handle, $L \subset S^3$ the attaching link of the 2-handles, then

$$\mathcal{S}_N^0(W^4; \emptyset) \cong \underline{\mathrm{KhR}}_N(L)$$

where $\underline{\mathrm{KhR}}_N(L)$ depends on KhR_N of cables of L.

E.g. $\dim_q \left(\mathcal{S}_N^0(S^2 \times D^2; \emptyset, \alpha)\right) = \prod_{k=1}^{N-1} \frac{1}{1-q^{2k}}$, results for $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$.