Knot homology and coherent sheaves on Coulomb branches

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My goal today: a purely algebraic perspective on Mina's talk.

In particular, she introduced:

- a space $\mathcal{X}$ as the Coulomb branch of a quiver gauge theory and
- an action of affine braids (in fact, affine tangles) on the derived category of T-equivariant coherent sheaves on $\mathcal{X}$.

I want to explain these notions in a way that doesn't require to know any physics or even very much algebraic geometry.

Key idea: $\mathcal{X}$ is a resolution of singularities of a singular variety $\mathcal{X}_0$. Replace them with a **non-commutative resolution**: a non-commutative algebra A with $D^b(A\text{-mod}) \cong D^b(\mathsf{Coh}(\mathcal{X}))$.

What's the deal with Coulomb branches? I mean specifically for 3d $\mathcal{N} = 4$ supersymmetric gauge theories for a group G and a matter representation N.

Mystery for a long time. Physicists would tell us if H is the Cartan of G, then

$$\mathfrak{M} \approx T^{*L}H/W$$

but there are "quantum corrections" that change this to a more complicated variety.

One hint: if $N = 0$, then $\mathfrak{M} = \mathrm{Spec}(H_*^{G[[t]]}(\mathrm{Gr}))$. Braverman, Finkelberg and Nakajima figured out how to make this hint precise.

In physics terms, this is computing the local operators of the A-twist of this theory as the endomorphisms of the trivial line operator.

We'll specialize to the case of a quiver gauge theory for a quiver Γ:

$$G = \prod_{i\in\Gamma} GL(v_i) \qquad N = \bigoplus_{i\to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_{i\in\Gamma} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

Taylor series $\mathsf{C} = \mathbb{C}[[t]]$	$\mathsf{G} = G[[t]]$	$\mathsf{N} = N[[t]]$
Laurent series $\mathcal{C} = \mathbb{C}((t))$	$\mathcal{G} = G((t))$	$\mathcal{N} = N((t))$

Relevant spaces:

$$\mathsf{Y} = \mathsf{N}/\mathsf{G} = \mathrm{Map}(D = \mathrm{Spec}\,\mathsf{C} \to N/G)$$

$$\mathcal{Y} = \mathcal{N}/\mathcal{G} = \mathrm{Map}(D^* = \mathrm{Spec}\,\mathcal{C} \to N/G)$$

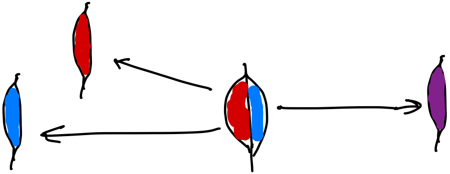
The spaces Y and $\mathcal{Y}$ are moduli spaces of framed quiver representations (FQR) with coefficients in the rings $\mathbb{C}[[t]] = \mathsf{C}$ and $\mathbb{C}((t)) = \mathcal{C}$. The map $\mathsf{Y} \to \mathcal{Y}$ is $\mathcal{C} \otimes_{\mathsf{C}} -$.

Recall that a **lattice** in $\mathcal{C}^n$ is an C-submodule isomorphic to C^n. The affine Grassmannian of GL_n is the space of lattices in $\mathcal{C}^n$.

Thus, choosing a preimage of $V_{\mathcal{C}} \in \mathcal{Y}$ under this map is choosing a lattice in $V_{\mathsf{C}} \subset V_{\mathcal{C}}$ which is invariant under the quiver representation maps.

The fiber product $\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}$ is the moduli space of FQR over $\mathcal{C}$ with a pair of compatible lattices.

Usual convolution arguments give a product on $\mathsf{A} = H_*^{BM}(\mathsf{Y} \times_{\mathsf{y}} \mathsf{Y}; \Bbbk)$.



This means we look at the double-raviolo, pullback a by pulling of raviolo 1, pullback b by pulling off raviolo 2, and the pushforward $a \cap b$ by mixing the fillings.

Definition

The **(3d) Coulomb branch** *is the spectrum* $\mathfrak{M} = \operatorname{Spec} \mathsf{A}$.

This geometric description is beautiful, but it's not very practical from an algebraic standpoint. An analogous situation is geometric construction of KLR algebras.

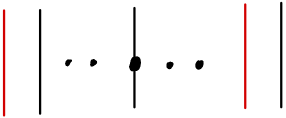
Let Y be the moduli space of FQR over $\mathbb{C}$. We'll be interested in a flag $F_1 \subset F_2 \subset \cdots$ of subrepresentations. Any such flag gives me a word in $I \cup I$ by looking at which node the dimension jumps on (red for framing nodes):

$$\dim(F_k \cap \mathbb{C}^{v_j}/F_{k-1} \cap \mathbb{C}^{v_j}) = \delta_{j,i_k}.$$

$$\dim(F_k \cap \mathbb{C}^{w_j}/F_{k-1} \cap \mathbb{C}^{w_j}) = \delta_{j,\mathsf{i}_k}.$$

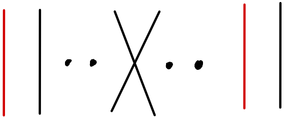
Theorem (W., Varagnolo-Vasserot, Rouquier)

We can write the KLRW algebra as $R_{\mathbf{v}} = H_*^{BM}(X \times_Y X)$.

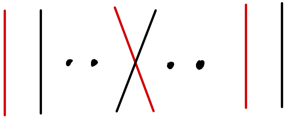


tautological Chern class

$c_1(F_{k+1}/F_k)$



pairs w/ $F_k = F_k'$ for $k \neq m$.

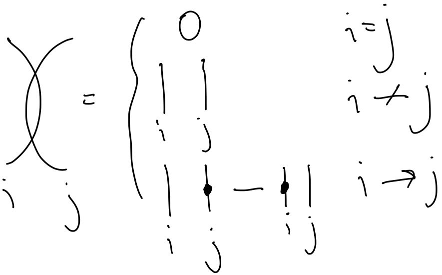


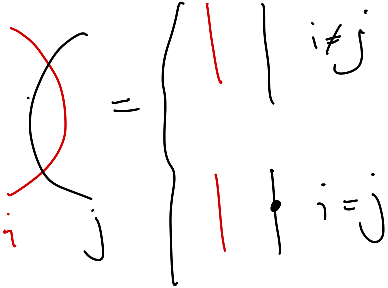
pairs w/ $F_k = F_k'$ $\forall k$

compatible w/ top + bottom

Theorem (W., Varagnolo-Vasserot, Rouquier)

We can write the KLRW algebra as $R_{\mathbf{v}} = H_*^{BM}(X \times_Y X)$.





How do we affinize this story?

Definition

An **affine flag** *in* $\mathbb{C}^m$ *is a sequence of a lattices* $\mathsf{F}_k \subset \mathbb{C}^m$ *for* $k \in \mathbb{Z}$ *such that*

$$\cdots \subset \mathsf{F}_k \subset \mathsf{F}_k \subset \mathsf{F}_{k+1} \subset \cdots \qquad t\mathsf{F}_k = \mathsf{F}_{k-m}$$

Objects describing affine flags are periodic (periodic permutations for Schubert cells, etc.)

We can look now at the moduli space of quiver representations that preserve an affine flag. We have to specify which nodes the jumps occur on.

A **periodic word** i is a map $\mathsf{i}\colon \mathbb{Z} \to I \cup I$ such that $\mathsf{i}_k = \mathsf{i}_{k+m}$ for all k for $m = \sum v_i + w_i$ such that any m consecutive entries contain v_i copies of i and w_i copies of i

Any homogeneous affine flag $\mathsf{F}_\bullet \subset \bigoplus_{i\in I} \mathbb{C}^{v_i} \oplus \mathbb{C}^{w_j}$ has a periodic word as its type, defined by

$$\dim(\mathsf{F}_k \cap \mathbb{C}^{v_j}/\mathsf{F}_{k-1} \cap \mathbb{C}^{v_j}) = \delta_{j,\mathsf{i}_k}.$$

$$\dim(\mathsf{F}_k \cap \mathbb{C}^{w_j}/\mathsf{F}_{k-1} \cap \mathbb{C}^{w_j}) = \delta_{j,\mathsf{i}_k}.$$

Let X_i be the moduli space of quiver reps over $\mathcal{C}$ together with a choice of affine flag of subreps of type i which matches the standard flag on $\mathcal{C}^{w_j}$.

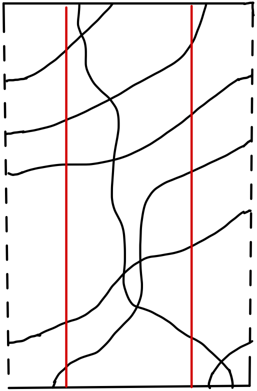
- In physics, this gives us a half-BPS vortex line operator compatible with the A-twist of the theory, where we couple to quantum mechanics on the flag variety, and think of preserving the flag as a restriction on the poles and zeros of fields.
- Mathematically, we can think of this as the D-module pushforward by the map $\mathsf{X}_\mathsf{i} \to \mathcal{Y}$.

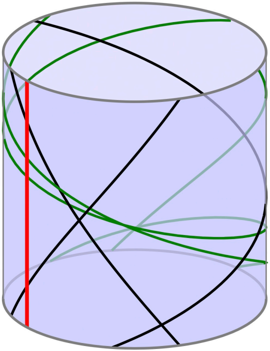
Thus we'll want to consider the convolution algebra

$$\mathsf{R} = \bigoplus_{\mathsf{i},\mathsf{j}} H_*^{BM}(\mathsf{X}_\mathsf{i} \times_{\mathcal{Y}} \mathsf{X}_\mathsf{j}) \cong \operatorname{Ext}^*\Big(\bigoplus_{\mathsf{i}} \pi_* \mathbb{C}_{X_\mathsf{i}}\Big).$$

Theorem

The convolution algebra R *has a presentation by cylindrical KLRW diagrams with the same local relations.*





Theorem

Assuming Γ is type ADE and w_i is only non-zero on minuscule nodes, the convolution algebra R *is a non-commutative symplectic resolution of* $\mathfrak{M}$ *for any ordering of the red strands.*

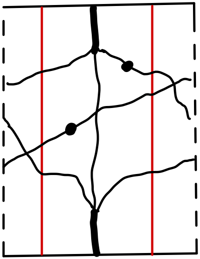
In fact, R is the endomorphisms of a tilting generator on any (usual) symplectic resolution $\mathcal{X}$ of $\mathfrak{M}$, which you can also construct using work of Bezrukavnikov and Kaledin in characteristic p.

In particular, there is an idempotent $e_0 \in \mathsf{R}$ such that

$$\mathsf{A} = \mathbb{C}[\mathcal{X}] = e_0 \mathsf{R} e_0.$$

The elements of this subalgebra can be thought of as pinching the strands at the top and bottom to a single thick strand (in the style of "thick calculus").

We're using the usual process of a taking a divided power ("one v_i!th") of the v_i strands with label i.



As discussed in Mina's talk, there is an action of cylindrical tangles on this category. This is intimately tied to a **real variation of t-structures**.

The (non-equivariant) **central charge** gives a function on $K(\mathsf{R}\text{-mod}) \to \mathbb{R}$ depending on the position of the red strands (the B-field) on the circle.

- For each set of labelled points $\mathbf{x}$ on the circle, we have an idempotent $e_{\mathbf{x}}$.
- The central charge of a module M is given by the integral $\mathcal{Z}(M) = \int \dim(e_{\mathbf{x}} M) d\mathbf{x}$

Both the action of crossings and of cups/caps are pinned down by this function, and in particular, its behavior as two red points collide.

When two red points pass through each other, we have equivalences

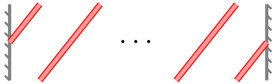
$$D^b(\mathsf{R}\text{-mod}) \cong D^b(\mathsf{Coh}(\mathcal{X})) \cong D^b(\mathsf{R}'\text{-mod})$$

Not unique: two most obvious possibilities correspond to the two ways points can swap in the space of complexified Kähler parameters.

Theorem

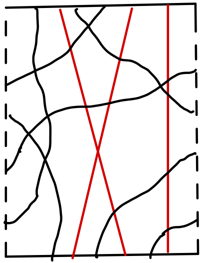
The resulting **wall-crossing functors** *generate an action of the affine braid groupoid (keeping track of labels on strands) acting on the categories* $D^b(\mathsf{R}\text{-mod})$ *for the different orders of red strands.*

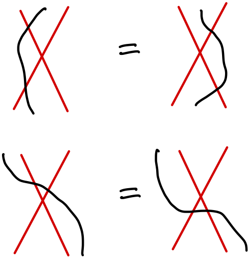
These are perverse equivalences by a theorem of Losev, with perverse filtration depending on order of vanishing of $\mathcal{Z}(M)$.



On purely abstract grounds, these functors come from tensor product with a complex of bimodules, but we can explicitly construct the corresponding bimodules:

They are cylindrical versions of the R-matrix bimodules for KLRW algebras.

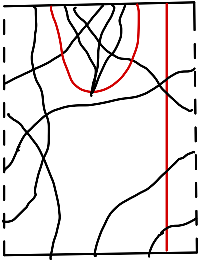


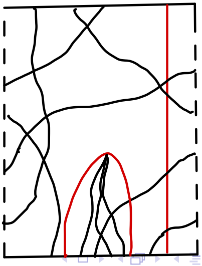


The cup functor is an equivalence between:

- Objects in R-fdmod with maximal vanishing order as red strands labeled with λ and $\lambda^* = -w_0\lambda$ come together.
- Objects in R′-fdmod for this algebra with the two red strands deleted (as well as $\alpha_i^\vee(\lambda+\lambda^*)$ black strands with label i).

Cap functor is biadjoint (up to shift). Realized by a cylindrical version of cup and cap functors for KLRW algebras:

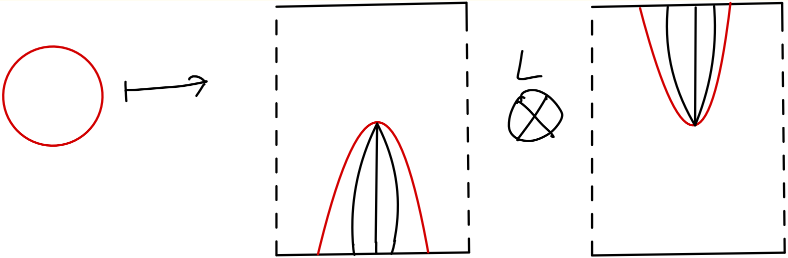




Theorem

These braid and cup/cap functors define a functor

- *from the category of oriented affine ribbon tangles, labeled with minuscule fundamentals,*
- *to the category of dg-categories with morphisms given by functors up to quasi-isomorphism.*



Making a labeled ribbon link annular in the boring way, this gives a link homology $\mathscr{D}_{coh}(K)$.

Theorem

The following link homologies are all the same:

- $\mathscr{D}_{coh}(K)$*, constructed from the affine tangle action above.*
- *the invariant constructed in my older knot homology work (which matches Khovanov-Rozansky in type A).*
- *Aganagić's physical construction.*

Of course, this gives an *annular* knot invariant as well.

Conjecture

In type A, this agrees with annular Khovanov-Rozansky homology (as defined by Queffelec and Rose).

The categories of $\mathsf{R}\text{-mod}$ for all possible labelings by fundamentals should carry an action of annular foams (by the web bimodules defined by Mackaay-W.)

This reduces to the check that a single unknot looped around the cylinder has the right value. I can do this calculation in $\mathfrak{sl}_2$, and am one ugly complex away from doing so in $\mathfrak{sl}_n$.

Thanks for listening.