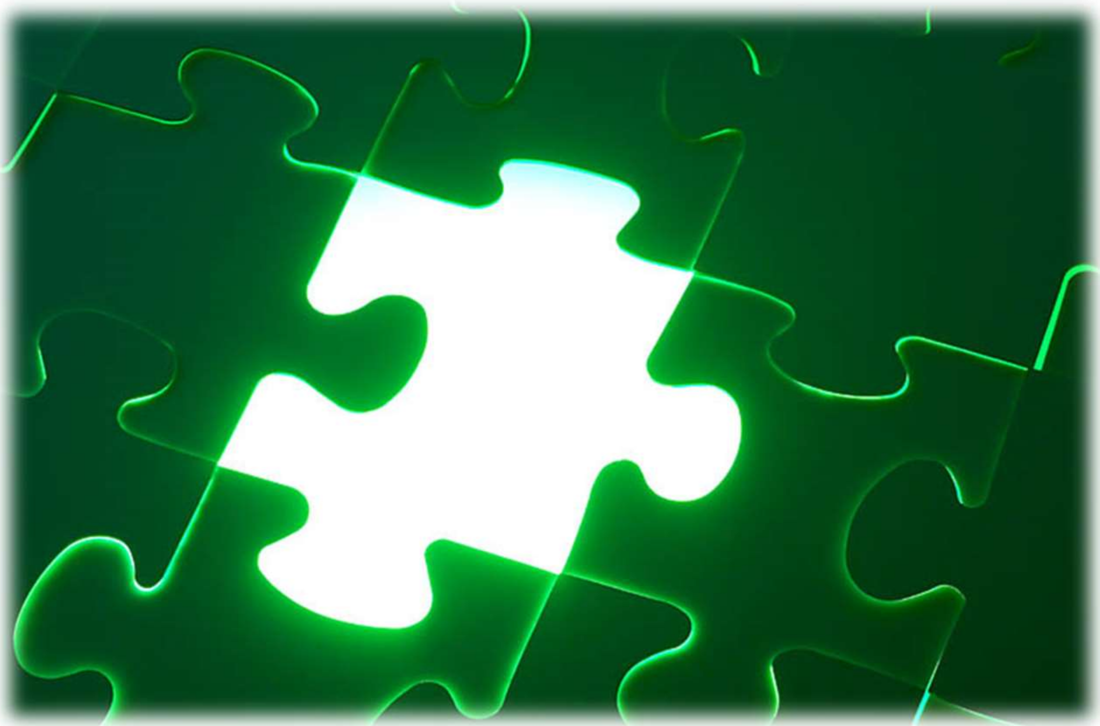


One peculiar feature (new structure) from multiple perspectives:

- Annular Khovanov homology — as in talks of Michael, Mina, Melissa, …
- Spectral sequences — cf. Eugene's talk
- Enumerative curve / BPS counting — as in Tobias' talk
- Kapustin-Witten equations — as in Edward's talk
- 3d-3d correspondence
- Rozansky-Witten theory based on $T^*\mathrm{Gr}_G$ — cf. Ben's talk
- Vertex algebras & quantum groups — as in Tudor's talk

Based on the spectacular success of the Khovanov homology, that categorifies the Jones polynomial,

$$J_K(q) = \sum_{i,j} (-1)^i q^j \ \dim Kh_{i,j}(K)$$

it is natural to ask whether Witten-Reshetikhin-Turaev (WRT) invariants of 3-manifolds admit a similar categorification:

$$\mathrm{WRT}(M_3; k) = \sum \ \ldots \ \dim H(M_3)$$

One immediate obstacle is that the WRT invariants, defined at roots of unity, do not come in the form of a polynomial / power series in $q = \exp(2\pi i/k)$ with integer coefficients, e.g.

$$\left(\frac{k}{2}\right)^{g-1} \sum_{j=1}^{k-1} \left(\sin \frac{\pi j}{k}\right)^{2-2g}$$

Possible ways around this challenge:

- Hopfological algebra — M.Khovanov, Y.Qi, A.Beliakova, …
- Higher representation theory — R.Rouquier, A.Manion, …
- Holomorphic q-series in |q|<1 — this talk

Surprise: multiple q-series

$$\widehat{Z}_b(M_3;q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(M_3;b)$$

S.G., P.Putrov, C.Vafa
S.G., D.Pei, P.Putrov, C.Vafa

labeled by $b \in H_1(M_3;\mathbb{Z}) \cong \mathrm{Spin}^c(M_3)$

S.G., C.Manolescu
S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

so that $\mathrm{WRT}(M_3,k) = \sum_b c_b^{\mathrm{WRT}} \widehat{Z}_b(q)\Big|_{q\to e^{\frac{2\pi i}{k}}}$

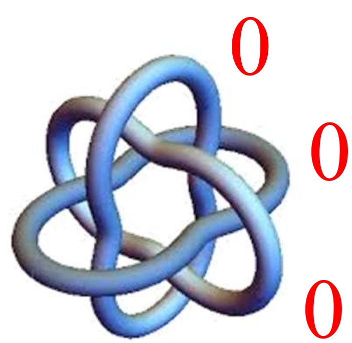
$M_3 = S^3_{1/r}(K)$	one q-series
$S^3_p(K)$	$b \in \mathbb{Z}_p$
	infinitely many:
$S^1 \times S^2$	$b \in \mathbb{Z}$
$S^1 \times \Sigma_g$	$b \in \mathbb{Z}^{2g+1}$
$\vdots$	$\vdots$

In the approach based on surgeries, one first needs to construct invariants for knot (or link) complements:

$$F_K(x,q) := \sum_{b\in\mathbb{Z}} x^b\, \widehat{Z}_b\left(S^3 \setminus K\right)$$

<u>Theorem [Lickorish, Wallace]:</u>
Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link in S^3.

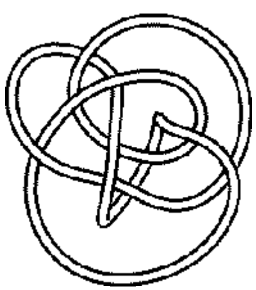
$$T^3 =$$



For knot and link complements, a very efficient diagrammatic approach based on the R-matrix for Verma modules and quantum groups at generic q was proposed by Park.

For example, using this approach and the GM surgery formula one finds:

$S^3_{+5}(\mathbf{10_{145}})$



$$
\begin{aligned}
b=2: &\quad q^{14/5}\left(-1+2q+2q^2+q^3+\ldots\right)\\
b=1: &\quad q^{11/5}\left(-1-2q^2-2q^3-4q^4+\ldots\right)\\
b=0: &\quad 2q^4+2q^7+2q^8+2q^9+4q^{10}+\ldots\\
b=-1: &\quad q^{11/5}\left(-1-2q^2-2q^3-4q^4+\ldots\right)\\
b=-2: &\quad q^{14/5}\left(-1+2q+2q^2+q^3+\ldots\right)
\end{aligned}
$$

One peculiar feature (new structure) from multiple perspectives:

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- Vertex algebras & quantum groups — as in Tudor's talk

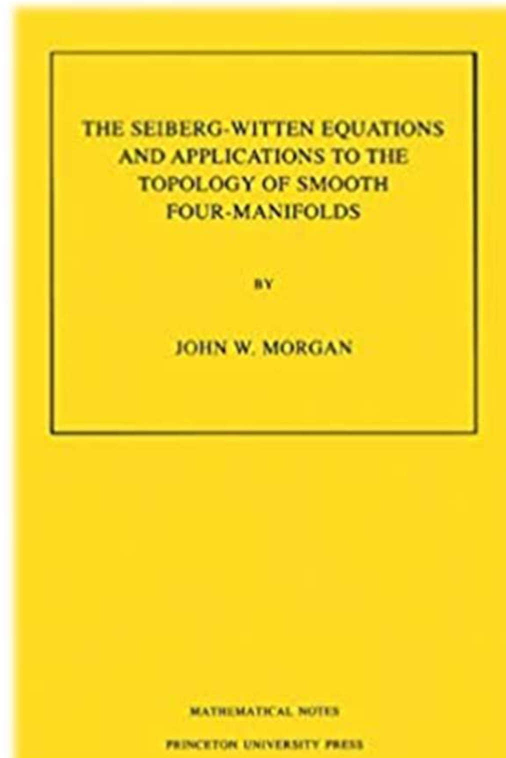
There is a canonical map:

$$\sigma : \mathrm{Spin}(M_3) \times H_1(M_3, \mathbb{Z}) \longrightarrow \mathrm{Spin}^c(M_3)$$

induced by

$$B\mathrm{Spin} \times BU(1) \to B\mathrm{Spin}^c$$

which, in turn, is part of the fiber sequence for the classifying spaces.



$$\Omega_3^{\mathrm{Spin}} = 0$$

Spectral sequences:

E_1

HOMFLY-PT homology

$\Longrightarrow KhR_{-N}(K)$

$\Longrightarrow$ “trivial”

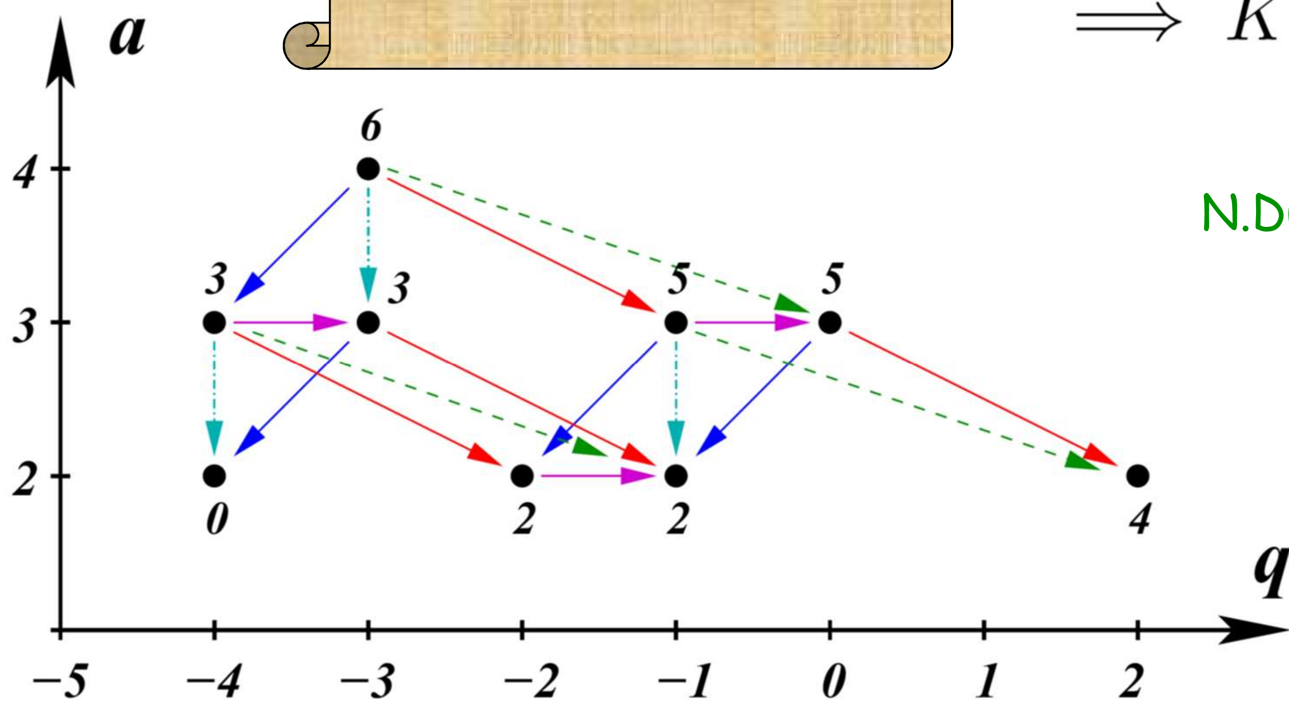
$\Longrightarrow HFK(K)$

$\Longrightarrow$ “trivial”

$\Longrightarrow Kh(K)$

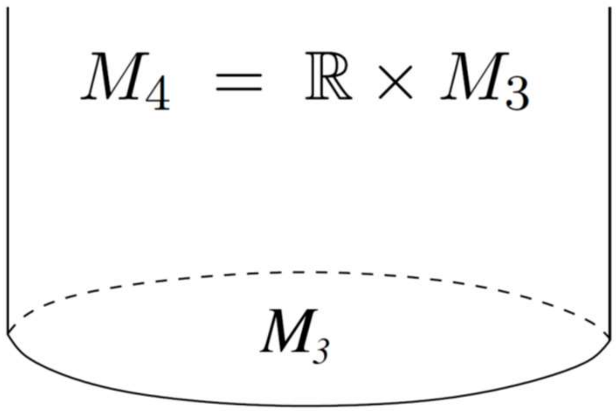
$\Longrightarrow KhR_N(K)$

E.Lee
B.Gornik
N.Dunfield, S.G., J.Rasmussen
J.Rasmussen
E.Gorsky, S.G., M.Stosic
⋮
C.Manolescu, M.Marengon
⋮



3-manifold version of knot Floer homology is the Heegaard Floer homology: requires a choice of Spin^c structure.

P.Ozsvath, Z.Szabo



Equivalent to:

- Monopole Floer homology — P.Kronheimer, T.Mrowka
- Embedded contact homology — C.Taubes, M.Hutchings

3-manifold version of knot Floer homology is the Heegaard Floer homology: requires a choice of Spin^c structure.

P.Ozsvath, Z.Szabo

correction terms: $d(M_3, b) \in \mathbb{Q}$

$b \in \mathrm{Spin}^c(M_3)$

➡ new proofs of Donaldson's diagonalization theorem and the Thom Conjecture

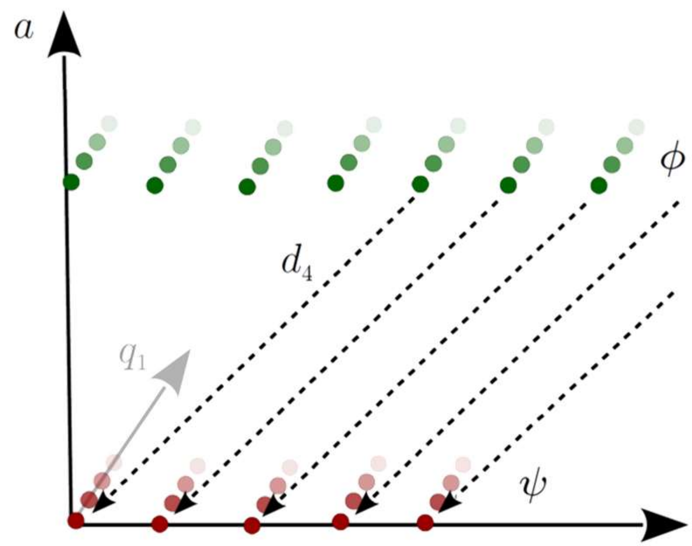
Example: $$d(L(p,1),b) = \frac{(p-2b)^2 - p}{4p}$$

$$b = 0, 1, \ldots, p-1$$

Choice of Spin^c structure is natural if Heegaard Floer homology and "Khovanov homology for 3-manifolds"

$$\widehat{Z}_b(M_3;q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(M_3;b)$$

unify in a larger framework, equipped with similar differentials.

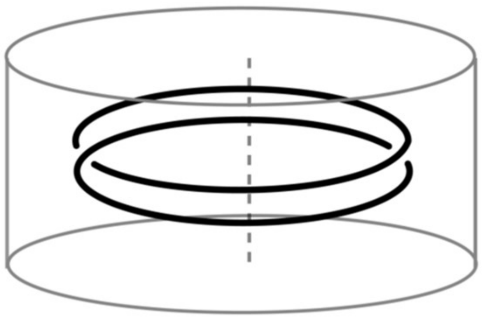


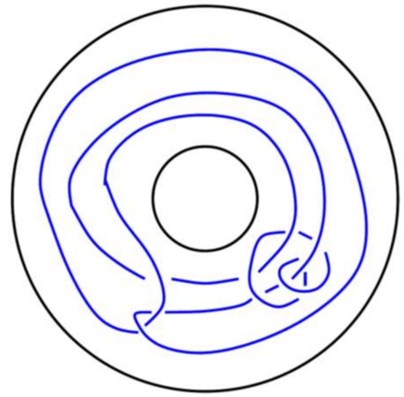
S.G., P.Putrov, C.Vafa

T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, P.Sulkowski

Annular Khovanov homology (a.k.a. sutured annular Khovanov homology):

M.Asaeda, J.Przytycki, A.Sikora





- triply-graded (homological, quantum, annular)

- spectral sequence

Spin^c

L.Roberts
E.Grigsby, S.Wehrli

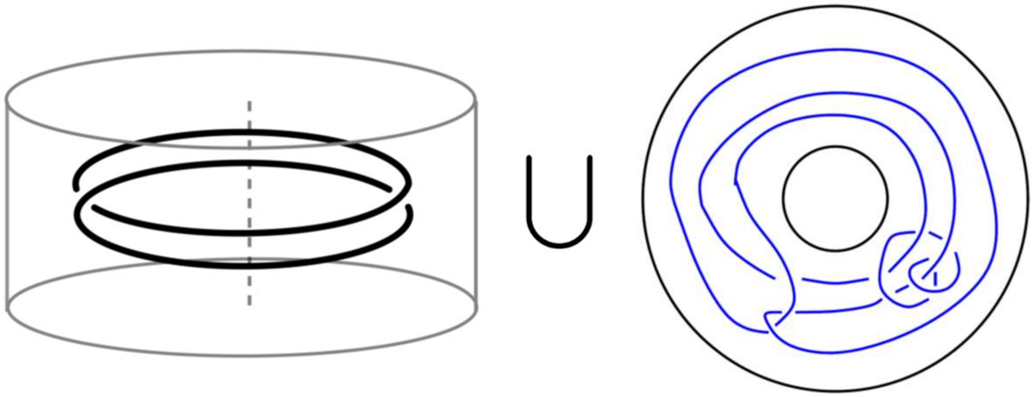
$$E^2 = AKh(\overline{K}) \implies E^\infty = SFH\left(\Sigma(A \times I, K)\right)$$

cf. $E^2 = Kh(\overline{K}) \implies E^\infty = \widehat{HF}\left(\Sigma(S^3, K)\right)$

P.Ozsvath, Z.Szabo

Annular Khovanov homology (a.k.a. sutured annular Khovanov homology):

M.Asaeda, J.Przytycki, A.Sikora



- carries an action of sl(2)

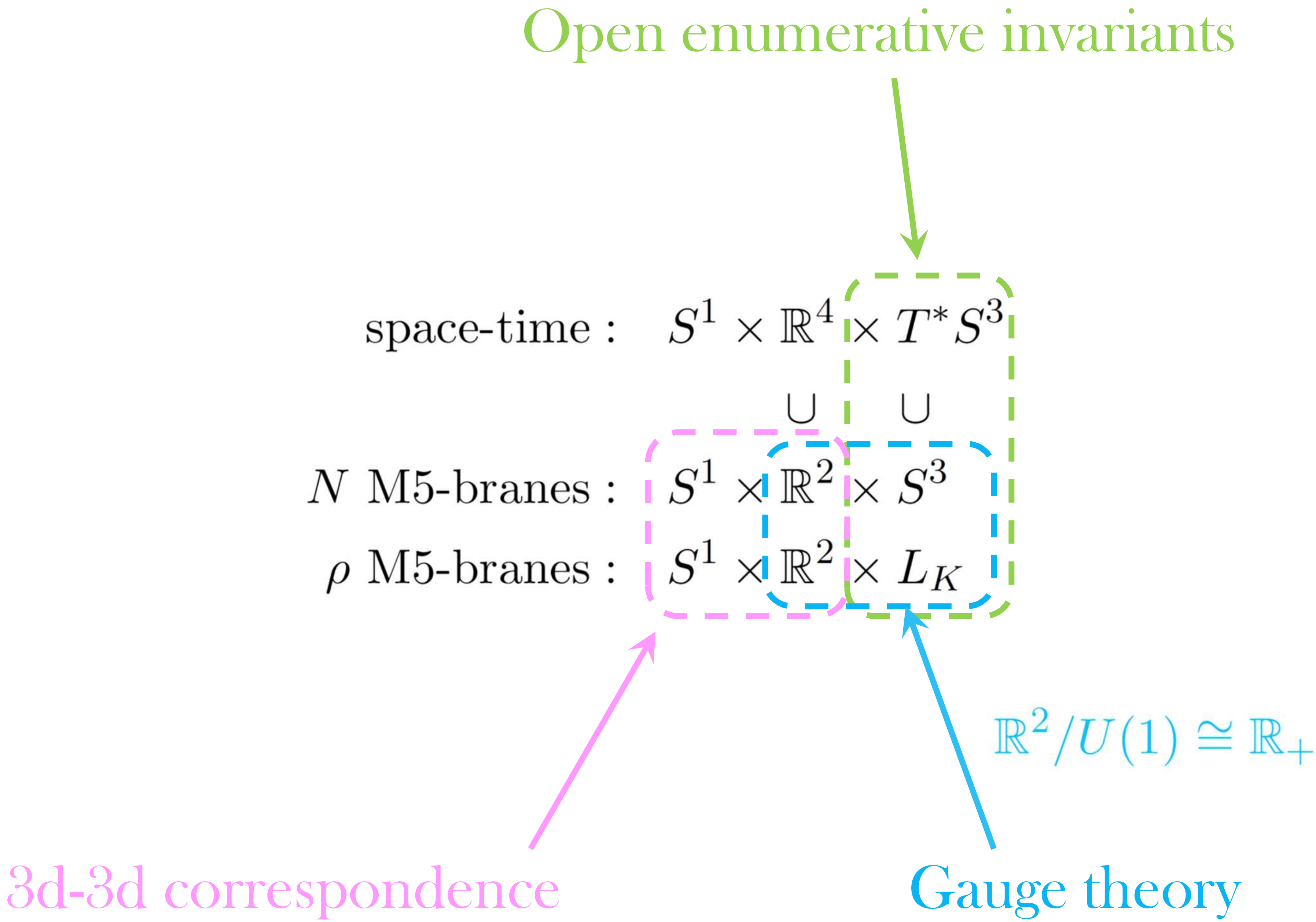
D.Auroux, E.Grigsby, S.Wehrli
A.Lauda
E.Grigsby, A.Licata, S.Wehrli

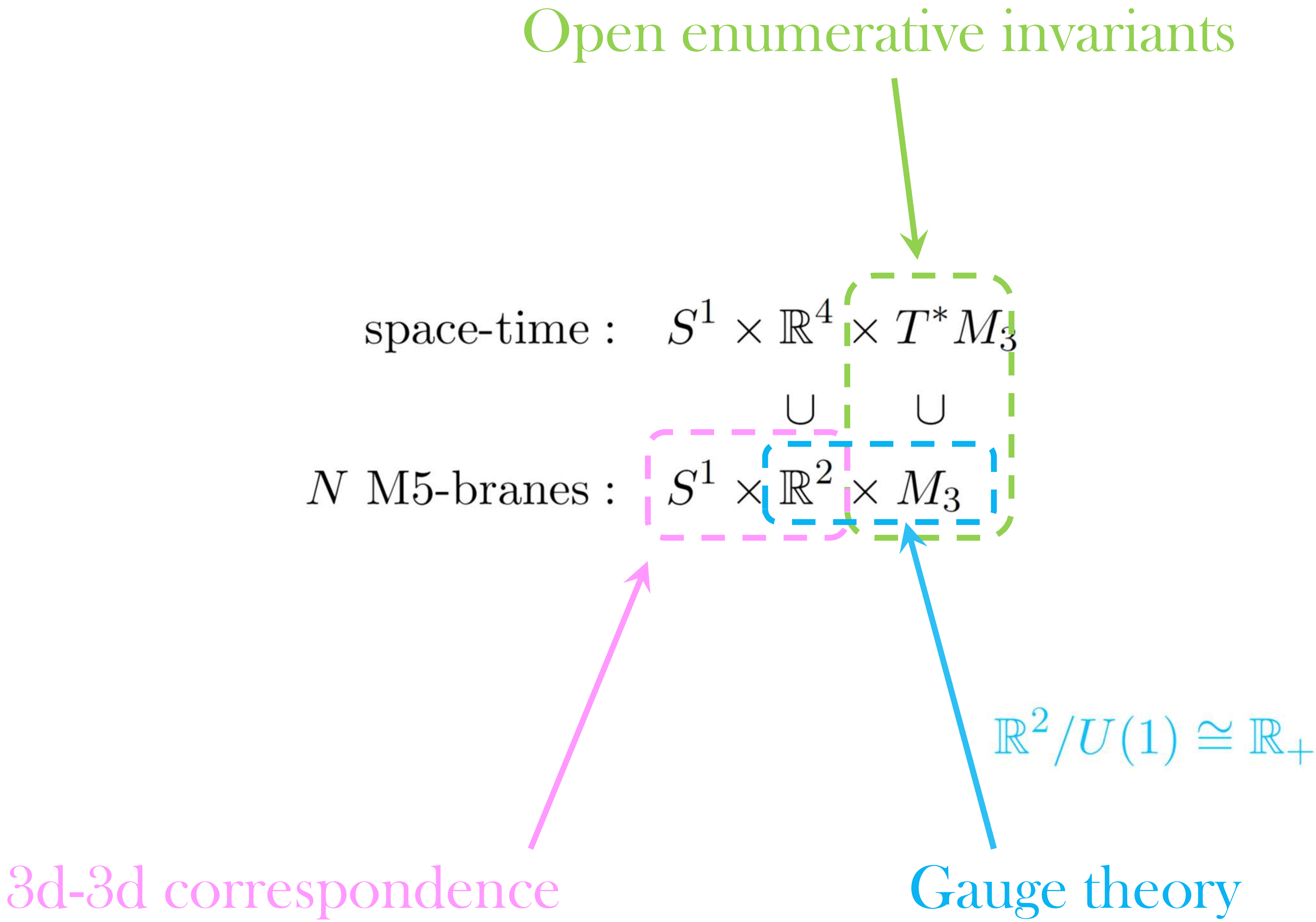
➡ b = sl(2) weight
and $x \in \mathrm{Hom}\big(H_1(M_3), \mathbb{C}^*\big)$ is a holonomy of a flat connection in the Cartan of SL(2,C)

- applying GM surgery formula

➡ agrees with $\widehat{Z}$ for simple links in $S^1 \times S^2$ and in Lens spaces

S.G., D.Pei, P.Putrov, C.Vafa


Open enumerative invariants
space-time : $S^1 \times \mathbb{R}^4 \times T^*S^3$
$\cup$ $\cup$
N M5-branes : $S^1 \times \mathbb{R}^2 \times S^3$
ρ M5-branes : $S^1 \times \mathbb{R}^2 \times L_K$
$\mathbb{R}^2/U(1) \cong \mathbb{R}_+$
3d-3d correspondence
Gauge theory

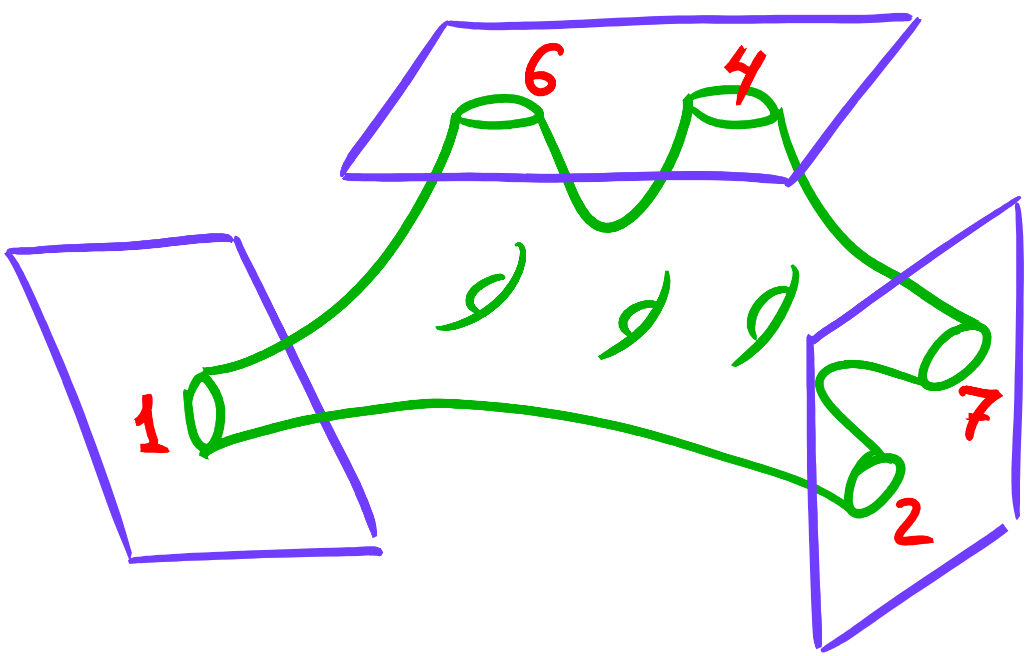


Open enumerative invariants
space-time : $S^1 \times \mathbb{R}^4 \times T^*M_3$
$\cup$ $\cup$
N M5-branes : $S^1 \times \mathbb{R}^2 \times M_3$
$\mathbb{R}^2/U(1) \cong \mathbb{R}_+$
3d-3d correspondence
Gauge theory

Open enumerative invariants:

$$\phi : (\Sigma, \partial\Sigma) \longrightarrow (X, L)$$

Σ genus g, with n boundary components



$$\partial\Sigma = \gamma_1 \sqcup \ldots \sqcup \gamma_n$$

$$\beta = \phi_*[\Sigma] \in H_2(X, L)$$

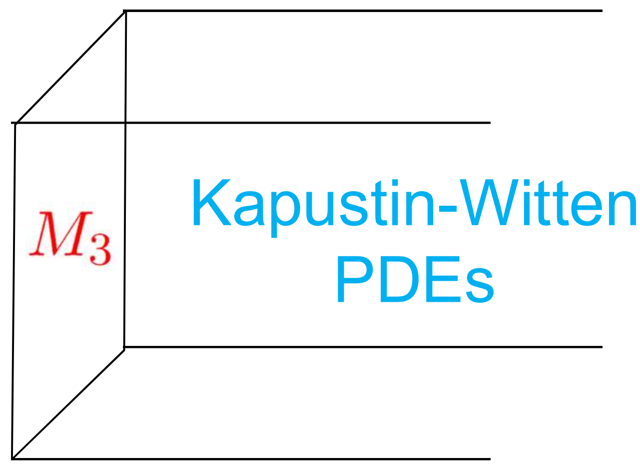
$$b_i = \phi_*[\gamma_i] \in H_1(L)$$

S.G., D.Pei, P.Putrov, C.Vafa

Gauge theory:

$\mathcal{B}_{\text{Nahm}}$

$$\frac{\sum_{i=1}^{3} e^i_\mu t_i}{y} + \dots$$



$\mathcal{B}_b$

e.g. $\mathbb{L} = U(1)$
and $x = e^{2\pi i \eta}$
when $M_3 = S^1 \times \Sigma$

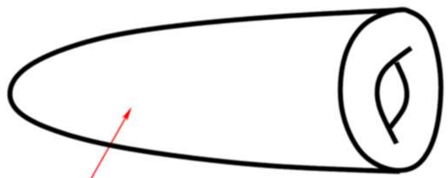
sometimes
requires a choice
of Spin structure

A version of the Freed-Witten
anomaly on the slab.

E.Witten
R.Mazzeo, E.Witten

cf. A.Gadde, S.G., P.Putrov

3d-3d correspondence:



b: background momentum/charge sectors of 2d boundary theory

M.Dedushenko, S.G., P.Putrov

$$\widehat{Z}_b(M_3;q) \xrightarrow{q \to \text{root of } 1} \text{CGP / ADO invariants}$$

$$\underset{\displaystyle \mathrm{Spin}^c(M_3)}{\widehat{Z}_{\underset{\cap}{b}}(M_3;q)} \xrightarrow{q \to 1} \frac{1}{\text{Turaev torsion}}$$

S.Chun, S.G., S.Park, N.Sopenko

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

$$F_K(x,q) := \sum_{b\in\mathbb{Z}} x^b \, \widehat{Z}_b\left(S^3 \setminus K\right) \xrightarrow{q\to 1} \frac{1}{\Delta_K(x)}$$

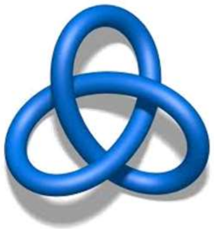
S.G., C.Manolescu

<u>Example:</u>

$$\eta(q) \ = \ \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}}$$

$$\frac{x^{-1}-x}{\Delta_{\mathbf{3_1}}(x^2)} = \sum_{m=1}^{\infty} \epsilon_m \left(x^m - x^{-m}\right)$$

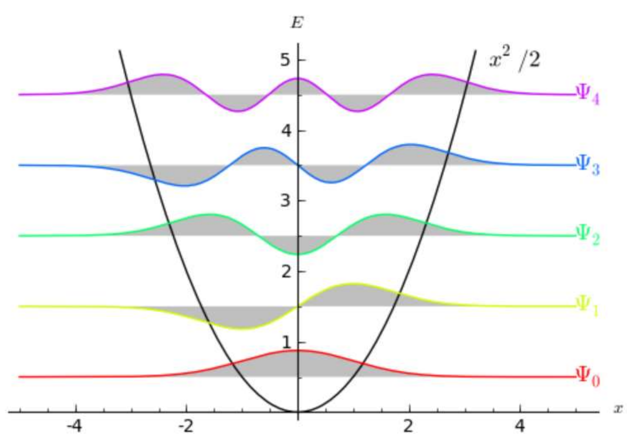
$$F_{\mathbf{3_1}}(x,q) = \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}} \left(x^m - x^{-m}\right)$$



Space of BPS states with q-degree bounded by n asymptotically grows as:

$$\dim H(K) \sim n^{\#}$$ 1d QM

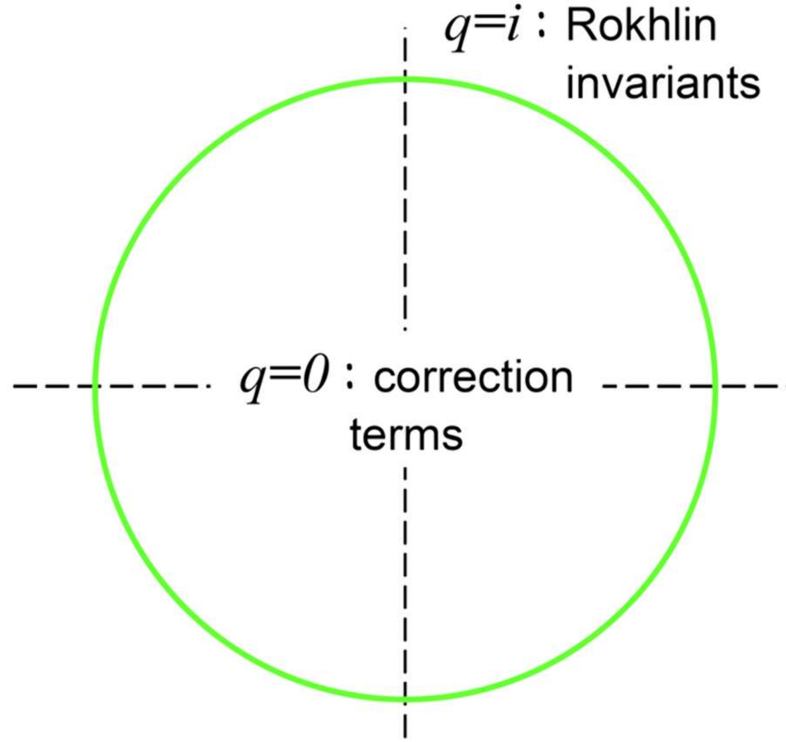
$$\dim H(M_3) \sim \exp\left(2\pi\sqrt{\frac{1}{6}c_{\text{eff}}\,n}\right)$$ 2d CFT



Relation to other 3-manifold invariants labeled by Spin or Spin^c structures?

$$\widehat{Z}_b(M_3, q)$$

$$\exp\left(-2\pi i \frac{3\mu(M_3,s)}{16}\right) = \sum_b c_{s,b}^{\text{Rokhlin}} \ \widehat{Z}_b(M_3,q)\Big|_{q=i}$$

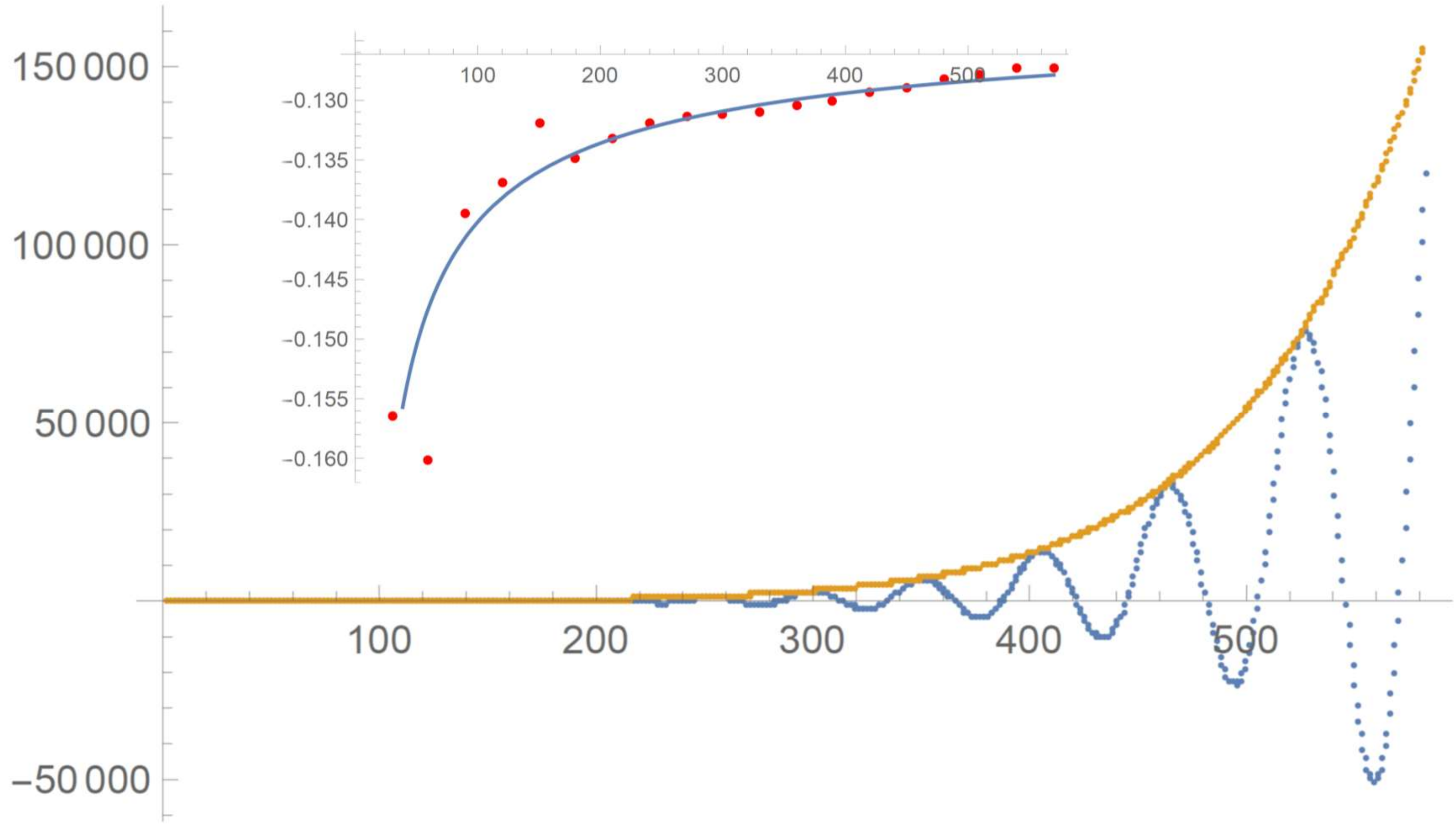


$$\Delta_b(M_3) = \frac{1}{2} - d(M_3,b) \mod 1$$

$$\widehat{Z}_b = q^{\Delta_b}\left(a_0^{(b)} + a_1^{(b)} q + a_2^{(b)} q^2 + \ldots\right) \ \in \ q^{\Delta_b}\mathbb{Z}[[q]]$$

$M_3 = S^3_{-1/2}($$)$:

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} + \ldots$$
$$\ldots - 15040q^{500} + \ldots)$$



cf. Neumann-Siebenmann invariant vs. correction terms:

A.Stipsicz
M.Ue
I.Dai

$$\overline{\mu}(M_3, s) \;=\; -4d(M_3, s)$$

and averaged versions:

$$\lambda(L(p,r)) \;=\; \frac{1}{p} \sum_{b \in \mathrm{Spin}^c(L(p,r))} d(L(p,r), b)$$

J.Rasmussen

$$\sum_{s \in \mathrm{Spin}(M_3)} \exp\left(-2\pi i \frac{3\mu(M_3,s)}{16}\right) = \sum_{s,b} c_{s,b}^{\mathrm{Rokhlin}} \; \widehat{Z}_b(M_3,q)\Big|_{q=i}$$

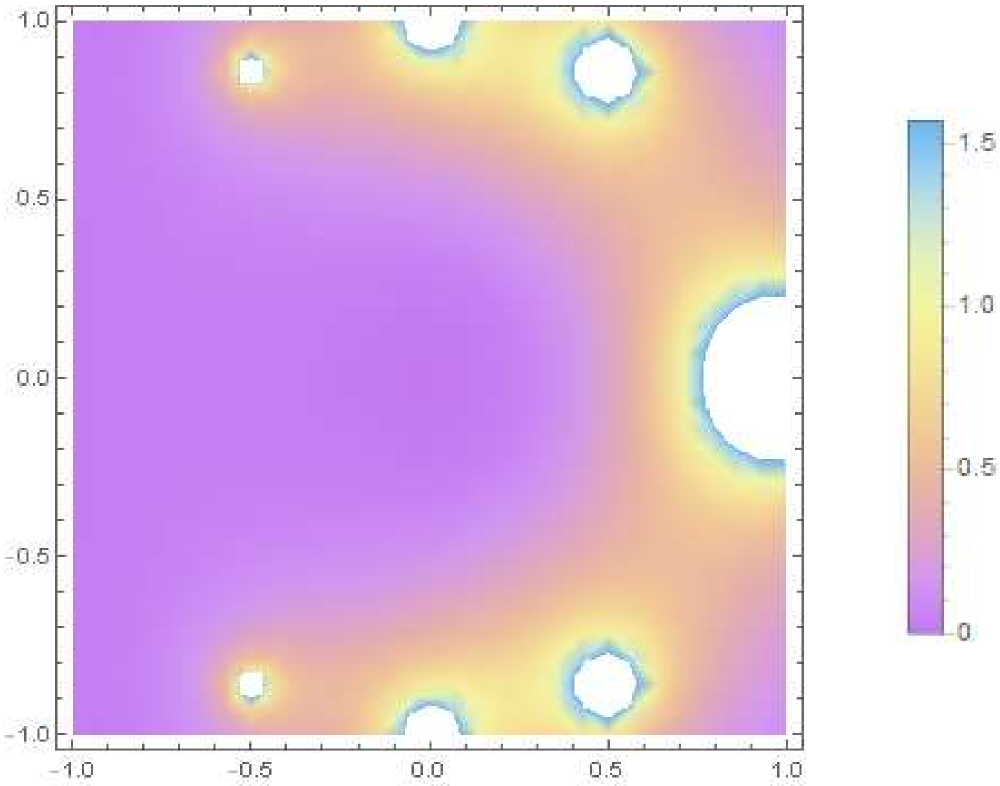
R.Kirby, P.Melvin

Question (about knots):

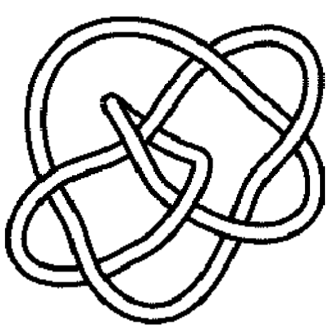
"quantum" ⟷ "homological"

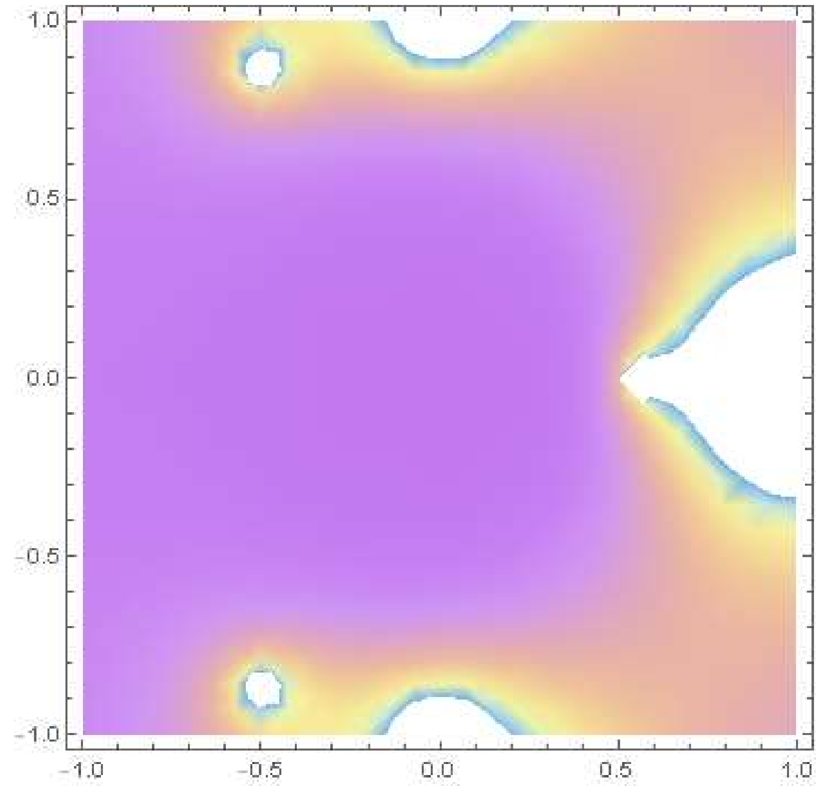
Jones,
HOMFLY-PT,
Kauffman, ...

s, s_n, τ, ν, ε, ν^+,
δ, δ_{p^n}, V_s, $\overline{V}_0$,
$\underline{V}_0$, $\Upsilon(t)$, ...

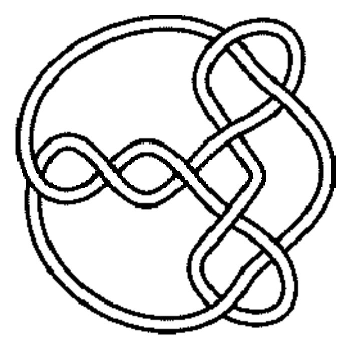


$$\left|J_{\mathbf{3_1}}(q) - J_{\mathbf{8_{10}}}(q)\right|^{-1}$$





$$\left|J_{\mathbf{4_1}}(q) - J_{\mathbf{9_{37}}}(q)\right|^{-1}$$



Thanks for listening.

Questions?