On the regularity of singular sets of minimizers for the Mumford-Shah energy

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Free Discontinuity Problems

Variational model in Image Segmentation and Edge Detection introduced by Mumford and Shah, CPAM '89: $g \in L^\infty(\Omega, [0,1])$ blurred image,
$u \in C^1(\Omega \setminus K)$, $K \subseteq \overline{\Omega} \subset \mathbb{R}^2$ compact.
A smoothed version of g is obtained by minimizing

$$(u, K) \to \mathcal{E}(u, K) + \alpha \int_{\Omega \setminus K} |u - g|^2 \, dx,$$

where

$$\mathcal{E}(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \mathcal{H}^1(K) < +\infty$$

Existence of minimizing couples

Main difficulty: find a topology on closed subsets of $\overline{\Omega}$ ensuring at the same time compactness of minimizing sequences and l.s.c. of $K \mapsto \mathcal{H}^1(K)$.

Two approaches:

- De Giorgi and Ambrosio's weak formulation thanks to the introduction of the $(G)SBV$ functional setting (Atti Accad. Naz. Lincei '88)
- Dal Maso, Morel and Solimini in 2d (Acta '92), Maddalena and Solimini in general (AIHP '01, ARMA '01) proved for K the so called uniform concentration property

In both cases Tonelli's Direct method then work.

Ahlfors regularity is a first mild regularity property of K: $\exists C \geq 1$ s.t.

$$C^{-1} r \leq \mathcal{H}^1(K \cap B_r(x)) \leq Cr$$

for all $x \in K$, $B_r(x) \subseteq \Omega$ (see De Giorgi, Carriero and Leaci, ARMA '89), Carriero and Leaci, Nonlinear Anal. '90)

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Regularity: scaling of the energy and Local Minimizers

(u, K) admissible couple on $B_r(x)$, set

$$u_r(y) = r^{-1/2} u(x + r\,y), \qquad K_r = r^{-1}(K - x)$$

then (u_r, K_r) admissible on B_1, and if u and $g \in L^\infty(\Omega, [0, 1])$

$$\frac{1}{r}\Big(\mathcal{E}(u, K, B_r(x)) + \int_{B_r(x)} |u - g|^2 dz\Big)$$
$$= \underbrace{\mathcal{E}(u_r, K_r, B_1)}_{=O(1)} + \underbrace{r^2 \int_{B_1} |u_r - g_r|^2 dy}_{=O(r)}$$

$\mathcal{M}(\Omega)$ is the class of *local minimizers*, i.e. if $\{v \neq u\} \cup (K \triangle J) \subset\subset \Omega$

$$\mathcal{E}(u, K) \leq \mathcal{E}(v, J)$$

Euler Lagrange equations I

$(u, K) \in \mathcal{M}(\Omega)$

- *Outer Variations*: $\forall \varphi \in C_c^1(\Omega)$

$$\int_{\Omega \setminus K} \nabla u \cdot \nabla \varphi \, dx = 0 \qquad \text{(OUT-VAR)}$$

- *Inner Variations*: $\forall \eta \in C_c^1(\Omega, \mathbb{R}^2)$

$$\int_{\Omega \setminus K} \left(|\nabla u|^2 \operatorname{div} \eta + 2\nabla^T u \cdot D\eta \cdot \nabla u \right) dx = - \int_K e^T \cdot D\eta \cdot e \, d\mathcal{H}^1 \qquad \text{(IN-VAR)}$$

$e : K \to \mathbb{S}^1$ Borel vector field tangent to K

Euler Lagrange equations II

$(u, K) \in \mathcal{M}(\Omega)$, in any open set A in which K is a smooth graph then

- *Outer Variations* equivalent to

$$\begin{cases} \triangle u = 0 & \text{on } A \setminus K \\ \partial_\nu u = 0 & \text{on } A \cap K \end{cases}$$

- *Inner Variations* equivalent to

$$\kappa = -|(\nabla u)^+|^2 + |(\nabla u)^-|^2 \qquad \text{on } A \cap K$$

$e : K \to \mathbb{S}^1$ Borel vector field tangent to $K \cap A$, κ curvature of $K \cap A$

Example of local minimizers

Alberti, Bouchitté and Dal Maso (Calc. Var. '03)

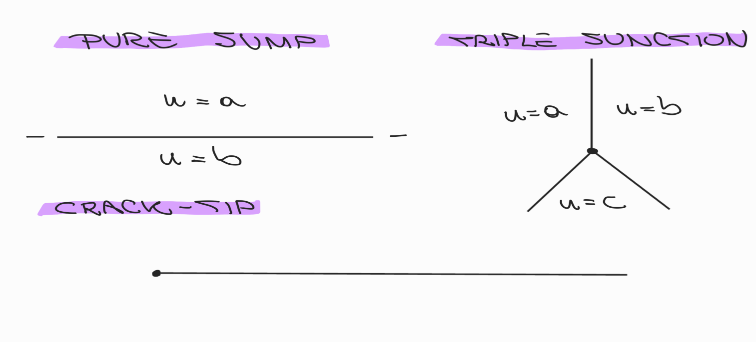
- Harmonic functions: $(u, \emptyset)$
- Pure Jump: u is locally constant on $B_r \setminus K$, and K is a diameter
- Triple Junction: u is locally constant on $B_r \setminus K$, and K is equal to three half lines meeting at equal angles in the origin (a propeller)

are local minimizers in $\Omega = B_r$ for r sufficiently small

Bonnet and David (Astérisque '01)

- Crack-tips: up to rotations, translations and addition of a constant $\sqrt{\frac{2}{\pi}\rho}\cos\left({}^{\theta}/_{2}\right)$, $\rho > 0$, $\theta \in (0, 2\pi)$, $K = [0, \infty) \times \{0\}$

...actually it is the only known (and conjectured) global minimizer!



The Mumford and Shah conjecture

Conjecture (Mumford and Shah, CPAM '89)

If $(u, K) \in \mathcal{M}(\Omega)$, $\Omega \subseteq \mathbb{R}^2$, *then* $\exists$ *(at most) countably many injective* C^1 *arcs* $\gamma_i : [a_i, b_i] \to \Omega$ *s.t.*

$$K = \cup_{i \in \mathbb{N}} \gamma_i([a_i, b_i])$$

(c1) *Any compact set* $E \subset \Omega$ *intersects at most finitely many arcs;*

(c2) *Two arcs can have at most an endpoint* p *in common, and if this is the case, then* p *is in fact the endpoint of three arcs, forming equal angles of* $^{2\pi}/_{3}$

If the conjecture holds, then K in $B_r(x)$, $x \in K$ and $r > 0$ small, is close to one among

(a) a diameter of $B_r(x)$

(b) a radius of $B_r(x)$

(c) a propeller centered in x, i.e. the union of three radii of $B_r(x)$ forming equal angles of $^{2\pi}/_{3}$

...Regularity theory establishes a partial strong converse!

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...Regularity theory establishes a partial strong converse!

ε-regularity theory

Theorem (Ambrosio, Fusco and Pallara, Ann. Sc. Norm. Pisa '97)

Let $(u, K) \in \mathcal{M}(\Omega)$, *then* $\exists\, \Sigma \subset K$ *relatively closed in* Ω *s.t.*

$$\mathcal{H}^1(\Sigma) = 0, \text{ and } \quad K \setminus \Sigma \text{ is locally a } C^{1,1} \text{ arc.}$$

Moreover, $\exists\, \varepsilon_0 > 0$ *s.t.*

$$\Sigma = \{x \in K : \liminf_{r \downarrow 0} \left(\mathscr{D}(x,r) + \mathscr{A}_\infty(x,r)\right) \geq \varepsilon_0\},$$

where

$$\mathscr{D}(x,r) = r^{-1} \int_{B_r(x)} |\nabla u|^2 dy$$

$$\mathscr{A}_\infty(x,r) = r^{-1} \min_{L \text{ line}} \sup_{K \cap B_r(x)} \mathrm{dist}(y, L)$$

In addition, in 2d David (SIAM'96) proved that $\dim_{\mathcal{H}} \Sigma < 1$.

ε-regularity theory

Theorem (Ambrosio, Fusco and Pallara, Ann. Sc. Norm. Pisa '97)
Let $(u, K) \in \mathcal{M}(\Omega)$, then $\exists\, \Sigma \subset K$ relatively closed in Ω s.t.

$$\mathcal{H}^1(\Sigma) = 0, \ \textit{and} \quad K \setminus \Sigma \textit{ is locally a } C^{1,1} \textit{ arc.}$$

Moreover, $\exists\, \varepsilon_0 > 0$ s.t.

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ε-regularity theory

$$\Sigma = \{x \in K : \liminf_{r\downarrow 0} (\mathscr{D}(x,r) + \mathscr{A}_\infty(x,r)) \geq \varepsilon_0\},$$

then

$$\Sigma = \Sigma^{(1)} \cup \Sigma^{(2)} \cup \Sigma^{(3)},$$

where

$$\Sigma^{(1)} = \{x \in \Sigma : \lim_{r\downarrow 0} \mathscr{D}(x,r) = 0\} \qquad (\textit{triple junctions})$$

$$\Sigma^{(2)} = \{x \in \Sigma : \lim_{r\downarrow 0} \mathscr{A}_\infty(x,r) = 0\} \qquad (\textit{crack-tips})$$

$$\Sigma^{(3)} = \{x \in \Sigma : \liminf_{r\downarrow 0} \mathscr{D}(x,r) > 0,\ \liminf_{r\downarrow 0} \mathscr{A}_\infty(x,r) > 0\}$$

according to the MS conjecture $\Sigma^{(3)} = \emptyset$

The set $\Sigma^{(1)} = \{x \in \Sigma : \lim_{r\downarrow 0} \mathscr{D}(x,r) = 0\}$

Theorem (David, SIAM '96)

$\exists\, \varepsilon > 0$, $c \in (0,1)$ s.t. if $(u,K) \in \mathcal{M}(\Omega)$, $z \in K$, $B_r(z) \subseteq \Omega$

$$r^{-1} \int_{B_r(z)} |\nabla u|^2 \, dx + r^{-1} \min_{P \text{ propeller}} \sup_{K \cap B_r(z)} \operatorname{dist}(y, P) < \varepsilon\,,$$

then $\exists\, \mathscr{C}$ a propeller, $\exists\, \Phi$ C^1-diffeomorphism s.t.

$$K \cap B_{cr}(z) = \Phi\,(\mathscr{C}) \cap B_{cr}(z)$$

Actually,

- the second summand is not needed in the ε-regularity criterion if $x \in \Sigma$
- $\Sigma^{(1)}$ is countable thanks to Moore's triod theorem

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The set $\Sigma^{(2)} = \{x \in \Sigma : \lim_{\rho\downarrow 0} \mathscr{A}_\infty(x,\rho) = 0\}$

Theorem (Bonnet and David, Astérisque '01)

$\forall \varepsilon_0 > 0\ \exists\, \varepsilon > 0$ s.t. if $(u,K) \in \mathcal{M}(\Omega)$ and

$$r^{-1}\mathrm{dist}_{\mathcal{H}}(K \cap B_r(z), \sigma) < \varepsilon$$

for some radius σ of $B_r(z) \subseteq \Omega$, then $\exists y_0 \in B_{r/4}(z)$ and some smooth $\gamma : (0, r/2) \to \mathbb{R}$ s.t.

$$K \cap B_{r/2}(z) = \big\{y_0 + \rho(\cos\gamma(\rho), \sin\gamma(\rho))\big\}$$

and

$$\sup_{(0,r/2)} \rho|\gamma'(\rho)| \le \varepsilon_0 \qquad \lim_{\rho\downarrow 0} \rho\,\gamma'(\rho) = 0$$

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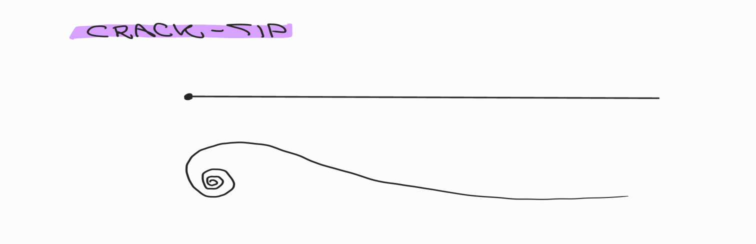
K might not be C^1 up to the tip!



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Theorem (Andersson and Mikayelyan preprint ArXiv '17,
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$\exists\, \varepsilon, \alpha > 0$ *s.t. if* $(u, K) \in \mathcal{M}(B_r(z))$ *with*

$$r^{-1}\mathrm{dist}_{\mathcal{H}}(K \cap B_r(z), \sigma) < \varepsilon$$

where $\sigma = z + r\mathrm{e}_1$, *then* $\exists y_0 \in B_{r/16}(z)$, $\psi \in C^{2,\alpha}([0, r/4], [0, r/8])$ *s.t.*

$$K \cap B_{r/4}(y_0) = \big\{y_0 + (t, \psi(t)) : t \in [0, r/4]\big\} \cap B_{r/4}(y_0)$$

and $\psi''(0^+) = 0$. *In particular, the curvature at the tip vanishes.*

Actually, it is true $\forall (u, K)$ critical point of the Mumford-Shah functional, i.e. s.t. (OUT-VAR) and (IN-VAR) hold, provided it is a smooth connected arc with an end-point in B_1

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Main ideas of the proof

(a) Harmonic conjugate
(b) Reparametrization
(c) Linearization
(d) Singular inner variations
(e) Decay properties of solutions to the linearized system and of the nonlinear one

Harmonic conjugate

(u, K) critical point in B_1, then by (OUT-VAR)

$$\int_{\Omega \setminus K} \nabla u \cdot \nabla \varphi \, dx = 0 \quad \forall \varphi \in C_c^1(B_1) \Longleftrightarrow \mathrm{curl}(\nabla u^{\perp}) = 0 \quad \mathcal{D}'(B_1)$$

Then $\exists w \in H^1_{loc}(B_1)$ s.t.

(i) w is harmonic on $B_1 \setminus K$, and $\nabla w = \nabla u^{\perp}$

(ii) $w \in C^{0,1/2}_{loc}(B_1)$ and
$$\sup_{x \neq y} \frac{|w(x) - w(y)|}{|x - y|^{\frac{1}{2}}} < \infty$$

(iii) w is constant on each connected component of K

(iv) w is unique up to addition of a constant

Reparametrization

Following L. Simon (Ann. Math., '83), for (u, K) critical point in B_1 set

$$\vartheta(t) := \gamma(e^{-t}),\ f(\phi, t) := e^{t/2} w\big(\phi + \vartheta(t), e^{-t}\big),\ \mathrm{isq}(\phi) := \sqrt{2/\pi}\,\sin(\phi/2)$$

Lemma

Then

$$\begin{cases} f_{tt} - f_t + \frac{f}{4} + f_{\phi\phi} + \big(\dot\vartheta f_\phi + \dot\vartheta^2 f_{\phi\phi} - 2\dot\vartheta f_{t\phi} - \ddot\vartheta f_\phi\big) = 0 \\ f(0,t) = f(2\pi, t) = 0 \\ \dfrac{\ddot\vartheta - \dot\vartheta - \dot\vartheta^3}{(1+\dot\vartheta^2)^{5/2}} = f_\phi^2(2\pi, t) - f_\phi^2(0,t) \end{cases} \qquad \text{(NON-LIN)}$$

and $\forall \sigma, \delta > 0$, $\forall k \in \mathbb{N}$ *if* ε_0 *is small enough*

$$\|\vartheta\|_{C^k([\sigma,\infty[)} + \|f - \mathrm{isq}\|_{C^k([0,2\pi]\times[\sigma,\infty))} \le \delta$$

Linearization

Theorem

Let $T > 0$, (u_j, K_j) be critical points in B_1 with $\gamma_j(1) = \vartheta_j(0) = 0$ and

$$\sup_{r\in(0,1]} (r|\gamma_j'(r)| + r^2|\gamma_j''(r)|) \sim \sup_{t>0}(|\dot{\vartheta}_j(t)| + |\ddot{\vartheta}_j(t)|) \leq \varepsilon_0(j) \downarrow 0$$

Set

$$\delta_j := \|f_j - \mathrm{isq}\|_{H^2([0,2\pi]\times[0,T])} + \|\dot{\vartheta}_j\|_{H^1([0,T])}$$
$$v_j(\phi, t) := \delta_j^{-1} f_j(\phi, t) \qquad \lambda_j(t) := \delta_j^{-1}\vartheta_j(t)$$

then, up to subsequences,

(a) *v_j converges weakly in $H^2([0, 2\pi] \times [0, T])$ and uniformly to some v;*

(b) *λ_j converges uniformly to some λ in $[0, T]$;*

(c) *the above convergences are in $C^{2,\beta}$ on $[0, 2\pi] \times [\sigma, T - \sigma]$ and $[\sigma, T - \sigma]$ respectively, $\forall \sigma \in (0, \frac{T}{2})$ and $\forall \beta \in (0, 1)$*

Moreover...

Linearization

...Moreover, the pair (v, λ) solves in $[0, 2\pi] \times [0, T]$

$$\begin{cases} v_{tt} - v_t + \frac{v}{4} + v_{\phi\phi} + (\dot{\lambda} - \ddot{\lambda})\mathrm{isq}_\phi = 0 \\ v(0, t) = v(2\pi, t) = 0 \\ \dot{\lambda}(t) - \ddot{\lambda}(t) = \sqrt{\frac{2}{\pi}}\,(v_\phi(0, t) + v_\phi(2\pi, t)) \\ \lambda(0) = 0 \end{cases} \tag{LIN}$$

and satisfies $\forall t \in (0, T)$

$$\int_0^{2\pi} \left[\left(\frac{v}{2} - v_t\right)(\phi, t)\left(\cos\frac{3\phi}{2} + \cos\frac{\phi}{2}\right) + v_\phi(\phi, t)\left(\sin\frac{3\phi}{2} + \sin\frac{\phi}{2}\right)\right] d\phi + \sqrt{\frac{\pi}{2}}\dot{\lambda}(t) = 0 \tag{VAR}$$

Linearization

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Singular inner variations

Theorem (De Lellis, F., Ghinassi, JMPA '21)

Let (u, K) be a critical point in B_1 and $y \in B_1$, then

$$\int_{B_r(y)\setminus K} (|\nabla u|^2 \operatorname{div}\eta - 2\nabla u^T \cdot D\eta \cdot \nabla u) + \int_{B_r(y)\cap K} e^T \cdot D\eta \cdot e \, d\mathcal{H}^1$$
$$= \int_{\partial B_r(y)\setminus K} \left(|\nabla u|^2 \eta \cdot \nu - 2\frac{\partial u}{\partial \nu} \eta \cdot \nabla u \right) d\mathcal{H}^1 + \sum_{x \in K \cap \partial B_r(y)} e(x) \cdot \eta(x)$$

for a.e. $r \in (0, 1 - |y|)$ and $\forall \eta \in C^1(\overline{B}_r, \mathbb{R}^2)$, where $\nu(x) = \frac{x-y}{|x-y|}$, e is tangent to K, $|e| = 1$ and $e(x) \cdot \nu(x) > 0$.

If $\eta(x) = x$, one gets back the David-Léger-Maddalena-Solimini formula.

Spectral analysis of the linearized system

$$v^e(\phi,t) := \frac{1}{2}(v(\phi,t) + v(2\pi-\phi,t))$$
$$v^o(\phi,t) := \frac{1}{2}(v(\phi,t) - v(2\pi-\phi,t))$$
$$\zeta(\phi,t) := v^o(\phi,t) - \lambda(t)\mathrm{isq}_\phi(t) = v^o(\phi,t) - \frac{\lambda(t)}{\sqrt{2\pi}}\cos\frac{\phi}{2}$$

Lemma

$(v,\lambda) \in H^2 \times H^3$ solves (LIN) iff $v^e, \zeta \in H^2$ are resp. even and odd s.t.

$$\begin{cases} v^e_{tt} - v^e_t + v^e_{\phi\phi} + \frac{1}{4}v^e = 0 \\ v^e(0,t) = v^e(2\pi,t) = 0 \end{cases}$$

$$\begin{cases} \zeta_{tt} - \zeta_t + \zeta_{\phi\phi} + \frac{1}{4}\zeta = 0 \\ \zeta_\phi(0,t) + \frac{\pi}{2}\left(\frac{1}{4}\zeta(0,t) + \zeta_{\phi\phi}(0,t)\right) = 0 \\ \zeta(0,0) = 0 \\ \zeta(0,t) = -\frac{\lambda(t)}{\sqrt{2\pi}} \end{cases}$$

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The linear three annuli property

Theorem

$\exists\, \eta > 0$, $\mathscr{L}(v, \lambda, a, b)$ s.t. $\forall\,(v, \lambda, a, b)$, $a < b$, solution of (LIN) and (VAR)

$$\mathscr{L}(v, \lambda, a, b) \sim \int_a^b \left(\|v(\cdot, t)\|^2_{H^2([0,2\pi])} + \dot{\lambda}^2(t) + \ddot{\lambda}^2(t) \right) dt$$

and

$$\mathscr{L}(v, \lambda, T, 2T) \geq (1-\eta)\mathscr{L}(v, \lambda, 0, T) \\ \Longrightarrow \mathscr{L}(v, \lambda, 2T, 3T) \geq (1+\eta)\mathscr{L}(v, \lambda, T, 2T)$$

The nonlinear three annuli property

Theorem

$\exists\, \delta > 0$, $\mathscr{L}(f,\vartheta,a,b)$ s.t. $\forall (f,\vartheta,a,b)$, $a < b$, solution of (NON-LIN)

$$\mathscr{L}(f,\vartheta,a,b) \sim \int_a^b \left(\|f(\cdot,t)\|^2_{H^2([0,2\pi])} + \dot{\vartheta}^2(t) + \ddot{\vartheta}^2(t) \right) dt$$

s.t.

$$\text{if } \|f - \mathrm{isq}\|_{H^2([0,2\pi]\times[kT,(k+1)T])} + \|\dot{\vartheta}\|_{H^1([kT,(k+1)T])} \le \delta,\ k \in \mathbb{N},$$

then

$$\begin{aligned} &\mathscr{L}(f,\vartheta,(k+1)T,(k+2)T) \ge (1 - \eta/2)\mathscr{L}(f,\vartheta,kT,(k+1)T) \\ &\Longrightarrow \mathscr{L}(f,\vartheta,(k+2)T,(k+3)T) \ge (1 + \eta/2)\mathscr{L}(f,\vartheta,(k+1)T,(k+2)T) \end{aligned}$$

where $\eta > 0$ is the constant of the linear three annuli property

Consequences of the nonlinear three annuli property

- either $\mathscr{L}(f, \vartheta, kT, (k+1)T) \leq (1 - \frac{\eta}{2})^k \mathscr{L}(f, \vartheta, 0, T) \qquad \forall k$
- or $\exists k_0$ s.t. $\mathscr{L}(f, \vartheta, kT, (k+1)T) \geq \mathscr{L}(f, \vartheta, k_0 T, (k_0+1)T)(1 + \frac{\eta}{2})^{k-k_0}$

The second alternative is however incompatible with the fact that

$$\lim_{k \uparrow \infty} \left(\|f - \mathrm{isq}\|_{H^2([0,2\pi]\times[kT,(k+1)T])} + \|\dot{\vartheta}\|_{H^1([kT,(k+1)T])} \right) = 0$$

$\Longrightarrow$

- $\exists T, \delta > 0$ s.t. $\|\dot{\vartheta}\|_{C^1([kT,(k+1)T])} \leq e^{-(1+\delta)k} \qquad \forall k$
- as $r = e^{-t}$ going back to γ: $|\kappa(r)| \leq Cr^\delta \qquad \forall r \in (0, 1]$

Thank you for your attention!