From Geometry to Arithmetic to Geometry

Geometry via Arithmetic workshop, BIRS, 12–16 July 2021

Jason Michael Starr
Report on joint work with
Zhiyu Tian (BICMR)

arXiv:1704.02932, arXiv:1811.02466 and arXiv:1907.07041

12 July 2021

Statement and Results.

Conjecture. James Ax.

Hypersurfaces $X_1, \dots, X_c$ in $\mathbb{P}^n_K$ with $\deg(X_1) + \cdots + \deg(X_c) \leq n$ contain a common geometrically irreducible K-variety.

János Kollár, char 0

True; even holds for all specializations of Fano manifolds.

Amit Hogadi and Chenyang Xu, char 0

True for all specializations of rationally connected varieties.

The proof uses MMP in char. 0.

Michael Fried and Moishe Jarden, char $p > 0$

True if $\deg(X_1)^2 + \cdots + \deg(X_c)^2 \leq n$ or if $K \supseteq \overline{\mathbb{F}}_p$.

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Geometric Proofs in Positive Characteristic.

Prime Regular DVR: Regular extension of a DVR with finite residue field (unramified, separable fraction field).

Separably rationally connected: $\exists f : \mathbb{P}^1_K \to X$, $f^* T_{X/K}$ ample.

S, $K \supseteq \overline{\mathbb{F}}_p$

X_R proper, flat over a prime regular DVR R, if geom. generic fiber is sep. rat. connected then base change by $R/\mathfrak{m} \to K$ has a geom. irred. K-subvariety.

Proof uses "RC Fibration Theorem" and Bertini's Connectedness Theorem.

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If geometric generic fiber is "rationally simply connected" then base change by $R/\mathfrak{m} \to K$ has a geom. irred. K-subvariety.

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Spaces of Rational Curves in Projective Space.

(Parameterized) Quasi-maps:
$\mathrm{QM}_{\mathbb{P}^1}(\mathbb{P}^n, e) = \mathbb{P}\mathrm{Hom}(H^0(\mathbb{P}^n, \mathcal{O}(1)), H^0(\mathbb{P}^1, \mathcal{O}(e))) \supset$
$\mathrm{Hom}((\mathbb{P}^1, \mathcal{O}(e)), (\mathbb{P}^n, \mathcal{O}(1)))$.

Unparameterized Quasi-maps: GIT quotient
$\mathrm{QM}_0(\mathbb{P}^n, e) := \mathrm{QM}_{\mathbb{P}^1}(\mathbb{P}^n, e)//\mathrm{Aut}(\mathbb{P}^1)$.

Stable maps: $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$, stack of degree e maps f from genus-0, at-worst-nodal curves with f-ample canonical bundle.

Quasi-map Contraction: Everywhere regular morphism
$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e) \to \mathrm{QM}_0(\mathbb{P}^n, e)$.

Projective target $X = \mathbf{Zero}(h_1, \dots, h_c) \subset \mathbb{P}^n$: $\mathrm{QM}_{\mathbb{P}^1}(X, e)$, resp. $\mathrm{QM}_0(X, e)$, $\overline{\mathcal{M}}_{0,0}(X, e)$ is the locus where pullbacks of $h_1, \dots, h_c$ vanish identically.

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First Irreducibility Theorems.

A. Kuznetsov, B. Kim – R. Pandharipande

Spaces of rational curves with fixed class are integral, smooth and have the "expected dimension" for $X = G/P$.

J. Harris – M. Roth – S, I. Coskun – S

General $X_d \subset \mathbb{P}^n$ with $d \leq (n+4)/2$, spaces are integral, LCI and have the "expected" dimension.

R. Beheshti – M. Kumar

Same for $d \leq (2n+2)/3$.

E. Riedl – D. Yang

Same for $d \leq n-2$, the optimal result.

Corollary

Genus-0 GW invariants are enumerative.

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New Irreducibility Theorems.

Tim Browning – Pankaj Vishe, Tim Browning – Will Sawin, char 0 or char $> d$

Smooth $X_d \subset \mathbb{P}^n$ with $(2d-1)2^{d-1} < n$ have $\mathrm{Hom}(\mathbb{P}^1, X)$ integral, LCI with "expected" dimension.

D. Testa proved irreducibility for del Pezzo surfaces of degree > 1. Using their formulation of the Geometric Manin Conjecture, there are results for Fano threefolds by Brian Lehmann – Sho Tanimoto. Combined with the Movable Bend and Break, there are newer results by Beheshti – Lehmann – Riedl – Tanimoto, Shimizu – Tanimoto, et al.

Browning-Vishe and Browning-Sawin follow a strategy of Jordan Ellenberg – Akshay Venkatesh via point-counting estimates of associated Hom spaces over finite fields, which are then proved by the Circle Method.

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Geometric Method.

S – Zhiyu Tian, char 0 or char $> d$

Smooth $X_d \subset \mathbb{P}^n$ with $d + \binom{2d+1}{d+1} < n$ have $\mathrm{QM}_{\mathbb{P}^1}(X, e)$ integral, ample complete intersection in $\mathrm{QM}_{\mathbb{P}^1}(\mathbb{P}^n, e)$ with "expected" dimension.

Above inequality is roughly $4^d/\sqrt{\pi d} < n$ compared to $d2^d < n$ in Browning–Vishe–Sawin.
Prithviraj Chowdhury extended this to complete intersections.

S, char 0 or char $> d$

Linear m-plane sections of X dominate moduli if $d + \binom{2d+1}{d+1} < n$.

Now set $m = d + 2$ so that the general m-plane section satisfies Riedl-Yang. Since we have integrality and "expected" dimension for the linear section $\mathrm{QM}_{\mathbb{P}^1}(\mathbb{P}^m, e)$ of $\mathrm{QM}_{\mathbb{P}^1}(X, e)$ in the projective space $\mathrm{QM}_{\mathbb{P}^1}(\mathbb{P}^n, e)$, every irreducible component of $\mathrm{QM}_{\mathbb{P}^1}(X, e)$ has the expected dimension and is integral.

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Conjecture and Lefschetz Hyperplane Theorem.

The spaces $\text{Hom}((\mathbb{P}^1, 0), (X, x_0), e)$ admit "glueing operations" by "glueing on a line" and deforming. Cohen-Jones-Segal study a stable limit (in homotopy theory) and its variation under "evaluation" to $x_0 \in X$.

Cohen – Jones – Segal Conjecture

For X Fano, if the stable limit with its evaluation to X is a quasifibration, then the stable limit is homotopic to the double loop space of X.

Original formulation due to Segal following his theorem for $X = \mathbb{P}^n$. Many cases proved for X "quasi-homogeneous". Cohen – Jones – Segal give a Floer theory heuristic, and use it to reprove the conjecture for $X = G/P$.
The homotopy type of X satisfies the Lefschetz hyperplane theorem: for $X \subset P$ an ample complete intersection, have isomorphism of homotopy groups until $\dim_{\mathbb{C}}(X)$.

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Singularities.

The closed complement $\mathrm{QM}_{\mathbb{P}^1}(X,e) \setminus \mathrm{Hom}(\mathbb{P}^1, X, e)$ is contained in the singular locus of $\mathrm{QM}_{\mathbb{P}^1}(X,e)$. So smooth "Purity Theorems" do not apply.

Grothendieck's "SGA2 Conjectures", proved by Hamm - Lê and sharpened by Goresky-MacPherson, do apply in homotopical degree $< c-1$ if the singular locus has codimension $\geq c$.

Browning–Sawin

For $n > 3(d-1)2^{d-1}$ the singular locus of $\mathrm{Hom}(\mathbb{P}^1, X, e)$ has codimension at least $\left(\frac{n}{2^{d-2}} - 6d + 6\right) \lfloor \frac{e+d}{d-1} \rfloor$.

S – Zhiyu Tian

For $c = 2b+3 \leq n-d$, for general X, the singular locus of $\mathrm{QM}_{\mathbb{P}^1}(X,e)$ and $\overline{\mathcal{M}}_{0,0}(X,e)$ have codimension $\geq c$ if $n \geq n_0 = d + b + (1/2) + \sqrt{d + b^2 + 5b + 2}$. Same for every smooth X if $n > n_0 + \binom{d+n_0-1}{n_0}$.

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Chris Skinner's Theorem.

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Weak approximation holds over global (number) fields for smooth X_d in $\mathbb{P}^n$ if $n > 2(d-1)2^{d-1}$.

The proof uses the Circle Method. The method seems to work for function fields if char $> d$.

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We also have a result over $\mathbb{F}_q(C)$ when n is greater than a doubly-exponential function in d using a variant of the Morin-Predonzan unirationality theorem.

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