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Heights and measures on Fano varieties

I Introduction

The initial problem may be described as follows

Let $F_1, \dots, F_m \in \mathbb{Z}[X_0, \dots, X_N]$ m homogeneous polynomials

Let V be the corresponding projective variety

$$V : F_i(X_0, \dots, X_N) = 0 \quad \text{for } 1 \leq i \leq m$$

We are interested in its rational points over $\mathbb{Q}$ that is in the set

$$V(\mathbb{Q}) = \{(x_0 : \dots : x_N) \in \mathbb{P}^N(\mathbb{Q}) \mid F_i(x_0, \dots, x_N) = 0 \text{ for } i = 1, \dots, m\}$$

One may ask several

Questions

1) Is $V(\mathbb{Q}) \neq \emptyset$? Is there a rational point

2) Is $V(\mathbb{Q})$ infinite ?

3) Is $V(\mathbb{Q})$ dense for Zariski topology in V?

4) If it is dense then "count" the points !

I am interested in the last question which I would like to explain in more details. To "count" the points one uses height function which more or less gives the "size" of a solution

height function $H : V(\mathbb{Q}) \longrightarrow \mathbb{R}_{>0}$

$$(x_0 : \dots : x_N) \longmapsto \sup_{0 \leq i \leq N} |x_i| \text{ if } \begin{cases} x_i \in \mathbb{Z} \\ \gcd_{0 \leq i \leq N} x_i = 1 \end{cases}$$

homogeneous coordinates

So the aim is to estimate the number of rational points of bounded height on V.

Aim

Describe the asymptotic behavior of

$$N_{V,H}(B) = \#\{x \in V(\mathbb{Q}) \mid H(x) \leq B\}$$

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Let me give you a few examples

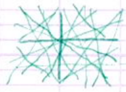
Example 1

For the projective space itself, it follows easily from Moebius inversion formula that

$$N_{\mathbb{P}^N_{\mathbb{Q}},H}(B) \underset{B\to+\infty}{\sim} 2^N \frac{1}{\zeta_{\mathbb{Q}}(N+1)} B^{N+1}$$

$$\frac{1}{\zeta_{\mathbb{Q}}(N+1)} = \prod_p \left(1-\frac{1}{p}\right)^{\mathrm{rk}\,\mathrm{Pic}\,V} \overbrace{\left(1+\frac{1}{p}+\cdots+\frac{1}{p^N}\right)}^{\#V(\mathbb{F}_p)/p^{\dim V}}$$

Let me illustrate this with a picture showing the points of height less than 30 on $\mathbb{P}^2(\mathbb{Q})$

Graphic n°1  Graphic n°2

Example 2

$V = \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ Of course there are many ways to see it as a subvariety of $\mathbb{P}^N$ so let us choose one

$$\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \longrightarrow \mathbb{P}^3_{\mathbb{Q}}$$

$$((x_0:x_1),(y_0:y_1)) \longmapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1)$$

The corresponding height is given by $H(x,y) = H(x)H(y)$ and it follows from the preceding case that

$$N_{V,H}(B) \underset{B\to+\infty}{\sim} \frac{8}{\zeta_{\mathbb{Q}}(2)^2} B^2 \log B$$

$$\frac{1}{\zeta_{\mathbb{Q}}(2)^2} = \prod \left(1-\frac{1}{p}\right)^{2\,(\mathrm{rk}\,\mathrm{Pic}\,V)} \left(1+\frac{1}{p}\right)^2 \quad \left(\left(1+\tfrac{1}{p}\right)^2 = \#V(\mathbb{F}_p)/p^2\right)$$

Here is the corresponding picture Graphic n°2 

Example 3

$V \xrightarrow{\pi} \mathbb{P}^2_{\mathbb{Q}}$ plane blown up in $P_0 = (0:0:1)$

The set of rational may be described as

$$V(\mathbb{Q}) = \{((x_0:x_1:x_2),(y_0:y_1)) \in \mathbb{P}^2(\mathbb{Q})\times\mathbb{P}^1(\mathbb{Q}) \mid y_0x_1 = y_1x_0\}$$

There again there are many way to embed in $\mathbb{P}^N$ but there is one which is more natural. Indeed $\omega_V^{-1} = \Lambda^{\dim V} T_V$ is very ample line bundle on V (the anticanonical line bundle) and induces an embedding $V \hookrightarrow \mathbb{P}^6_{\mathbb{Q}}$ and the corresponding height is given by

$$H((x_0:x_1:x_2),(y_0:y_1)) = \sup(x_0,x_1,x_2)^2 \sup(y_0,y_1)$$

if $x_i, y_i \in \mathbb{Z}$, $\gcd(x_0,x_1,x_2) = \gcd(y_0,y_1) = 1$

Let $E = \pi^{-1}(P_0)$ the exceptional divisor

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U its complement $U = V - E$

Then the result is as follows

Proposition

The number of points on the open subset is given by

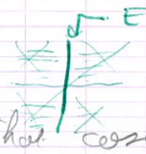
$$N_{U,H}(B) \underset{B\to+\infty}{\sim} \frac{8}{6} \prod_{p \text{ prime}} \overbrace{\left(1-\frac{1}{p}\right)^{2}}^{\text{rk Pic } V} \underbrace{\left(1+\frac{2}{p}+\frac{1}{p^2}\right)}_{\# V(\mathbb{F}_p)/p^2} B \log B$$

and on the exceptional divisor

$$N_{E,H}(B) \underset{B\to+\infty}{\sim} 2/\zeta_{\mathbb{Q}}(2) \; B^2.$$

Rem

Thus we have $N_{U,H}(B) = \overset{\text{negligible}}{o}\left(N_{E,H}(B)\right)$

which is clear on the picture Graphic n°3 

E contains most of the points. Thus in that case the dominant term in the asymptotic behavior of the number of points on the whole of V reflects the geometry of the line E only and the global geometry of V is "hidden". One of the basic idea of Manin is to say If one removes E and look at the rational points on U, this reflects the geometry of V. So one has to restrict oneselves to open subsets. So now what kind of asymptotic behavior do we get?

In all cases I know in which if $N_{U,H}(B) \to +\infty$

$$N_{U,H}(B) \sim C B^{a} (\log B)^{b-1} \text{ with } C>0,\ \overset{\in\mathbb{Q}}{a} \geq 0,\ b \in \tfrac{1}{2}\mathbb{Z},\ b \geq$$

So now the question is

Aim

Describe, a, b and C

If V is smooth and ω_V^{-1} ample (ie V Fano),

Manin gave a conjectural interpretation of a and b in terms of $L = \mathcal{O}_V(1) \in \operatorname{Pic} V$. In particular b should be maximal if $L \in \mathbb{R}_{>0}\, \omega_V^{-1}$

And in that case the asymptotic behavior should be of the form

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Empiric formula

If V smooth, $\mathcal{O}_V(1) = \omega_V^{-1}$ and $V(\mathbb{Q})$ is Zariski dense then, in general, for any small enough U, $\exists\, C > 0$ st

$$N_{U,H}(B) \underset{B \to +\infty}{\sim} C\, B (\log B)^{t-1} \quad ?$$

where $t = \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}\, V$

Rem.

In fact, there is a counter-example (Batyrev - Tschinkel) that is the reason for which I wrote "in general"

From now on V is smooth projective variety

$\mathcal{O}_V(1) = \omega_V^{-1} \Rightarrow V$ is Fano

Let us look at the constant C ⊛

For those of you who know about Tamagawa measures this strongly suggest that such a measure plays a role here. If you do not know about these Tamagawa measures there is nevertheless a good reason to look for a measure.

Rem

$V(\mathbb{Q}) \subset V(\mathbb{R}) \supset W$ open for the usual topology

$\#\{x \in V(\mathbb{Q}) \cap W \mid H(x) \leq B\}$?

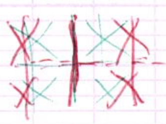
More generally

p prime $\left|\frac{a}{b}\right|_p = p^{v_p(b) - v_p(a)}$ absolute value on $\mathbb{Q}$.

$\mathbb{Q}_p$ completion.

$V(\mathbb{Q}) \subset V(\mathbb{Q}_p) \supset W$ open   same question.

[eg $\#\left\{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{Z}) \;\middle|\; \begin{cases} x_0 \equiv 0 \ (2) \\ H(x_0, x_1, x_2) \leq 30 \end{cases}\right\}$

Graphic n°4 ]

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It is in fact convenient to consider the space of Adeles

$V(\mathbb{A}_\mathbb{Q}) = \prod_{v \in M_\mathbb{Q}} V(\mathbb{Q}_v)$ $\quad M_\mathbb{Q} = \{\text{primes}\} \cup \{\infty\}$, $\mathbb{Q}_\infty = \mathbb{R}$

and to consider the measure

$$\mu_{H \leq B} = \frac{1}{N_{U,H}(B)} \sum_{\substack{x \in U(\mathbb{Q}) \\ H(x) \leq B}} \underbrace{\delta_{(x)}}_{\text{Dirac measure}}$$ probability measure on $V(\mathbb{A}_\mathbb{Q})$

Aim

construct measure ω_H on $V(\mathbb{A}_\mathbb{Q})$ ⟶ "limit" of $\mu_{H \leq B}$
⟶ constant C

II Construction of the measure.

The best way to do it is to introduce metrics on the line bundle ω_V^{-1}

$V \hookrightarrow \mathbb{P}^N_F$ with $\omega_V^{-1} = \mathcal{O}_V(1)$ this corresponds to $N+1$ section of ω_V^{-1} $s_0, \dots, s_N \in \Gamma(V, \omega_V^{-1})$

$\forall v \in M_\mathbb{Q}$ one defines $\|.\|_v : \omega_V^{-1}(\mathbb{Q}_v) \to \mathbb{R}$ continuous for v-adic topology

$$\begin{array}{c} \omega_V^{-1}(\mathbb{Q}_v) \\ \downarrow \text{ fiber of dim } 1 \\ V(\mathbb{Q}_v) \end{array}$$

$\forall x \in V(\mathbb{Q}_v), \forall s \in \Gamma(V, \omega_V^{-1})$ $\|s(x)\|_v = \inf_{\{i \mid s_i(x) \neq 0\}} \left| \frac{s(x)}{s_i(x)} \right|_v$

Using these metrics the height has a very simple expression

$\forall x \in V(\mathbb{Q})$ if $s \in \Gamma(V, \omega_V^{-1})$, $s(x) \neq 0 \Rightarrow H(x) = \prod_{v \in M_\mathbb{Q}} \|s(x)\|_v^{-1}$

Then one can define measures

ω_v measure on $V(\mathbb{Q}_v)$ given locally by

$$\omega_v = \left\| \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \right\|_v dx_{1,v} \dots dx_{n,v}$$

(cf. the way to produce a measure from a section of ω_V in real topology).

Here $x_1, \dots, x_n$ coordinate system on V

$dx_{i,v}$ Haar measure on $\mathbb{Q}_v$ (locally compact field)

For the real place it is something well know

At finite good places we do not get somethin new

$\int_{\mathbb{Z}_p} dx_p = 1$ $\quad dx_{i,\infty}([0,1]) = 1$

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Prop (Weil, P.)

For allmost all p (that is all exept a finite numb.)

$$\omega_p(V(\mathbb{Q}_p)) = \frac{\# V(\mathbb{F}_p)}{p^{\dim V}}$$

Pb We want a mesure on the space of adele, but $\prod_p \omega_p(V(\mathbb{Q}_p))$ diverges.

Thus one has to introduce

Converging factors

$\infty \in S \subset M_{\mathbb{Q}}$ finite, bad primes

$\overline{\mathbb{Q}}$ alg closure of $\mathbb{Q}$, $\overline{V} = V \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \overline{\mathbb{Q}}$

$\forall p \in M_{\mathbb{Q}} - S$, $L_p(s, \operatorname{Pic} \overline{V}) = \dfrac{1}{\operatorname{Det}(1 - p^{-s} \operatorname{Fr}_p \mid \operatorname{Pic} V_{\overline{\mathbb{F}_p}} \otimes \mathbb{Q})}$

(where Fr_p is induced by $x \mapsto x^p$ in $\overline{\mathbb{F}_p}$)

$$L_S(s, \operatorname{Pic} \overline{V}) = \prod_{p \in M_{\mathbb{Q}} - S} L_p(s, \operatorname{Pic} \overline{V})$$

$$\lambda_v = \begin{cases} L_v(1, \operatorname{Pic} \overline{V}) & \text{if } v \in M_{\mathbb{Q}} - S \\ 1 & \text{if } v \in S \end{cases}$$

Def of the measure

$$\underline{\omega}_H = \left[\lim_{s \to 1} (s-1)^t L_S(s, \operatorname{Pic} \overline{V})\right] \prod_{v \in M_{\mathbb{Q}}} \lambda_v^{-1} \omega_v$$

measure on $V(\mathbb{A}_{\mathbb{Q}})$.

For the constant one need one more rational factor

$\underline{C_{\mathrm{eff}}(V)} \subset \operatorname{Pic} V \otimes_{\mathbb{Z}} \mathbb{R}$

cone generated by the classes of effective divisors

$\underline{\alpha(V)} = \operatorname{Vol} \left\{ y \in (\operatorname{Pic} V \otimes_{\mathbb{Z}} \mathbb{R})^{\vee} \;\middle|\; \begin{array}{l} \forall x \in C_{\mathrm{eff}}(V), \langle x, y \rangle > 0 \\ \langle x, \omega_V^{-1} \rangle = 1 \end{array} \right\}$

$\in \mathbb{Q}$ in the known cases

$\beta(V) = \# H^1(\mathbb{Q}, \operatorname{Pic} \overline{V})$

Def of the conjectural constant

$$\theta_H(V) = \alpha(V)\, \beta(V)\, \underline{\omega}_H(\overline{V(\mathbb{Q})})$$

closure in $V(\mathbb{A}_{\mathbb{Q}})$

This gives us two empiric formulae

Empiric formulae V as above, $U(\mathbb{Q})$ Zariski dense

For small enough U

(F) $N_{U,H}(B) \underset{B\to+\infty}{\sim} \Theta_H(V)\, B\, (\log B)^{t-1}$?

(E) if $f : V(\mathbb{A}_\mathbb{Q}) \to \mathbb{R}$ continuous

$$\int_{V(\mathbb{A}_\mathbb{Q})} f \,\mathcal{N}_{H\le B} \underset{B\to+\infty}{\longrightarrow} \int_{V(\mathbb{A})} f \,\frac{\omega_H}{\omega_H(\overline{V(\mathbb{Q})})} \quad ?$$

III Results

(F)+(E) are

- compatible with Π of variety (Franke-Manin-Tschinkel)
- compatible with the results of the circle method

[(In particular true if V smooth, dim $N-m$ defined by m equations of deg d in $N+1$ variables with

$$N > 2^{d-1} m(m+1)(d-1)$$

(Birch))]

- true for flag varieties $V = G/P$ parabolic (In particular true for any quadric) (FMT, P.) ~~parabolic~~

(F) is

- true for toric varieties (ie equivariant compactification of alg. tori) (P., Robbiani in particular cases, Batyrev-Tschinkel in general)
- true for equivariant compactification of affine space. (Chambert-Loir, Tschinkel)
- true for the plane blown up in 4 rational points (Salberger, de la Bretèche)

...

False for

$V \subset \mathbb{P}^3_F \times \mathbb{P}^3_F$ defined by $\sum_{i=0}^{3} x_i y_i^3 = 0$ in $\mathbb{P}^3 \times \mathbb{P}^3$

(Batyrev-Tschinkel). the power of logarithm is too big

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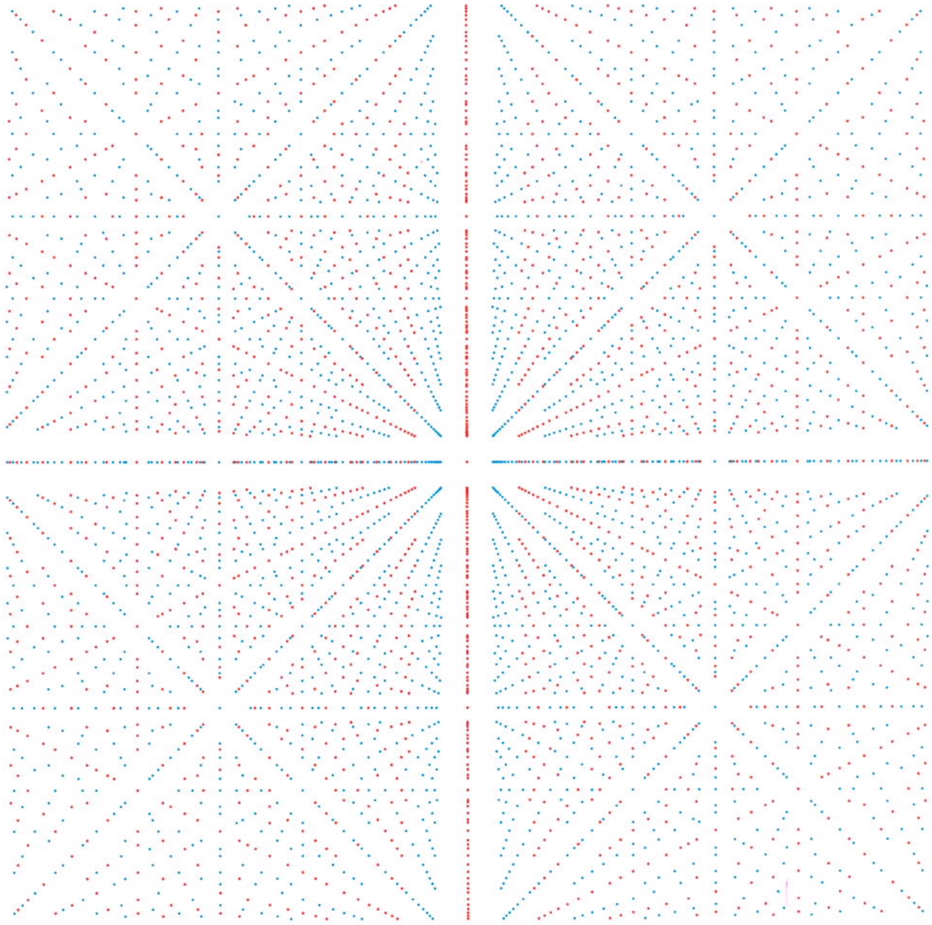
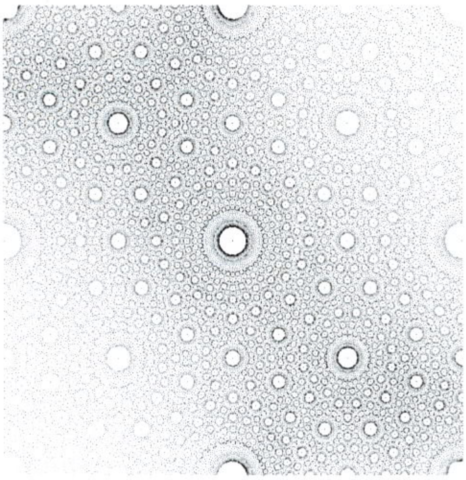
$\mathbf{P}^2$

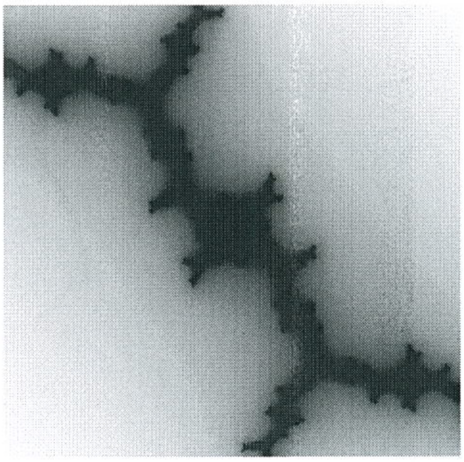


Set of $(a : b : c)$ in $\mathbf{P}^2(\mathbf{Q})$ such that
$\mathrm{gcd}(a, b, c) = 1, a\%2 = 0,$
$|a/c| < 1$, $|b/c| < 1$, $H((a : b : c)) < 30$,

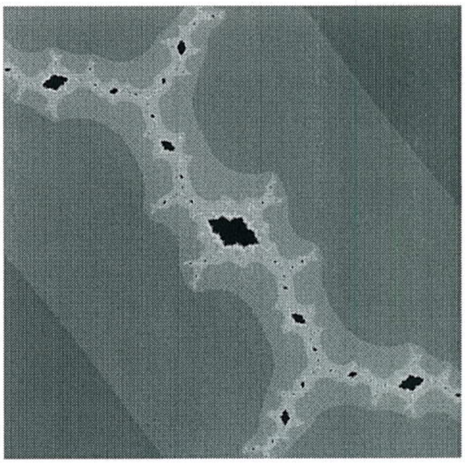


Points of bounded height

Density ρ



Function ρ^3



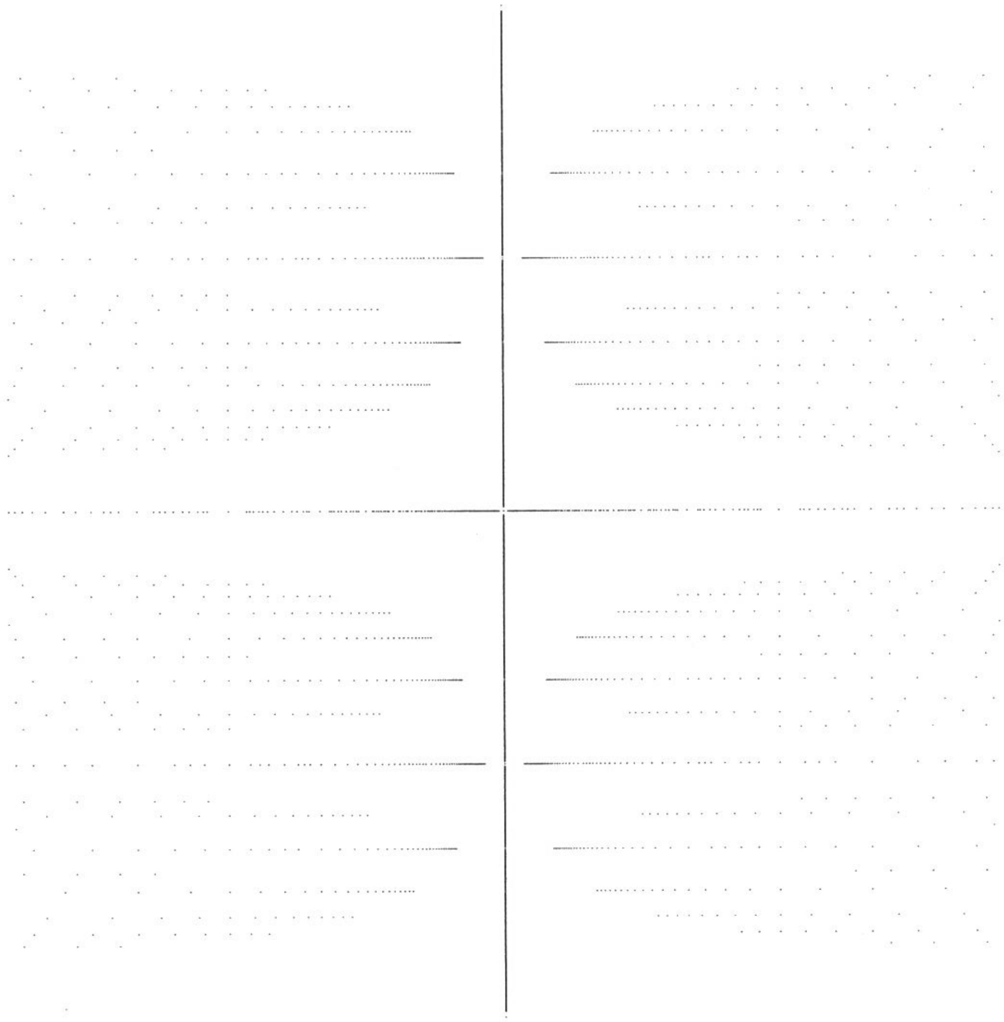
Julia set

Height corresponding to the dynamical system

$$(z_0 : z_1) \mapsto (z_0^2 + cz_1^2 : z_1^2)$$

with $c = \frac{-135+868i}{839}$

$V = \mathbf{P}^2$ blown up in a point



$V \subset \mathbf{P}^2 \times \mathbf{P}^1$

Set of $x = ((y_0 : y_1 : 1), (1 : z_1))$

in $V(\mathbf{Q})$, such that $|y_0| < 1$, $|z_1| < 1$, $H_{4,2}(x) < 16000$.