A locally conformally symplectic Weinstein conjecture

Weinstein conjecture

A Reeb vector field on a closed contact man. has Reeb orbits.

As lcs manifolds generalize contact manifolds, in some senses, does the Weinstein conjecture generalize to lcs setting?

Why not generalize Arnold conjecture, as lcs manifolds also generalize symplectic manifolds?

Issue: If $M = (C \times S^1, \omega)$ is the lcs-fication of a contact manifold (C, λ), then the Reeb vector field on C induces Hamiltonian transformations of M with no fixed pts.

Ex $S^3 \times S^1$, $\omega = d\lambda - d\theta \wedge \lambda$

$H = -1$, X_H solves:

$$\omega(X_H, \cdot) = d^\alpha H$$
$$= \cancel{dH} - \alpha \wedge H = \alpha$$

So $X_H = (R_\lambda, 0)$ check:

$$\omega(X_H, \cdot) = \lambda \wedge \alpha(X_H, \cdot)$$
$$= \alpha(\cdot) - \lambda(\cdot)\,\cancel{\alpha(X_H)}$$
$$= \alpha$$

Setup for the formulation

(M,ω) exact closed lcs mfld.

so $\omega = d^\alpha \lambda = d\lambda - \alpha \wedge \lambda$

ω non-degenerate

define a distribution $V \subset TM$

$V(p) = \ker d\lambda(p)$

each $V(p)$ has dim 2 or 0

cannot be identically 0

since then $d\lambda$ is non-degenerate, which is impossible.

Example $\omega = d\lambda - \alpha \wedge \lambda$ on $C \times S^1$

λ - contact form on C

$V(p) = \{R_\lambda(p), \frac{\partial}{\partial\theta}(p)\}$ span

Define a cone

$C \subset V$

$C = \{ v \in V \mid \lambda(v) > 0 \}$

Reeb curve in M

A smooth map $\sigma : S^1 \to M$

$\forall t : \dot\sigma(t) \in C$.

i.e. is tangent to C.

CSW conjecture

M closed, $\dim M \geq 4$

$\omega = d^\alpha \lambda$ an lcs structure

on M with α integral.

Then there is a Reeb curve in M.

CSW

Implies the Weinstein conjecture

(C,λ) contact, closed

set $(M = C\times S^1, \omega)$ to be the

lcs-fication of (C,λ)

If $o : S^1 \to M$ is a Reeb

curve and $\pi : M \to C$

the projection, then

$\pi(o) : S^1 \to C$ is a Reeb

orbit of C up to

parametrization.

First step

Hopf lcs structure

The lcs-fication of the standard contact form on S^{2k+1}.

Thm 1: CSW conjecture holds for a C^3 neighborhood of the Hopf lcs structure on $M = S^{2k+1} \times S^1$.

Proof is via holomorphic curves

Let (M, λ, α) be an exact lcs.

Definition of admissible almost complex structures on (M, λ, α)

$\mathcal{V} = \ker d^{\alpha}\lambda \subset TM$ as before

$\xi = d\lambda$ orthogonal complement to $\mathcal{V}$

J is admissible if:

- $J(\xi) \subset \xi$, $J(\mathcal{V}) \subset \mathcal{V}$
- J tames $d\lambda$ on ξ

Example: $M = C \times S^1$ is the lcs-fication of (C, λ)

Then $V(p) = \underbrace{\left\{ R_\lambda(p), \frac{\partial}{\partial\theta}(p) \right\}}_{\text{span}}$

$$\underbrace{\xi(p) = \xi_\lambda(p)}_{\text{contact distribution}} \oplus 0$$

Let J be admissible with $J(R) = \frac{\partial}{\partial\theta}$

If $\gamma : S^1 \to C$ is a λ-Reeb orbit, then

$$\boxed{\begin{aligned} & u_0 : T^2 \to M \\ & u_0(s,t) = (\gamma(s), t) \end{aligned}}$$

is J-holomorphic for a unique complex structure on T^2, satisfying

$$j\left(\frac{\partial}{\partial s}\right) = c\frac{\partial}{\partial t}$$

for c, s.t. $\dot{\gamma}(s) = c R_\lambda(\gamma(s))$.

The map u_0 is called a $\underline{\text{Reeb torus}}$.

So we obtain a map

R : "Reeb orbits" $/S^1 \to \mathcal{M}^{ell}$

$\mathcal{M}^{ell}$ - moduli space of elliptic J-curves with charge $(1,0)$.

charge: for fixed generators η, β of $H_1(T^2, \mathbb{Z})$

$\langle u^*\alpha, \eta\rangle = 1$, $\langle u^*\alpha, \beta\rangle = 0$

Lemma: R is bijective

Strategy for the proof theorem 1.

0) note that virtual dim of $\mathcal{M}^{ell}$ is 0.

1) Compute $GW = \pm\infty$

$\uparrow$

"counts" ellements of $\mathcal{M}^{ell}$. Since we don't have energy bounds count can be infinite.

2) Conclude that a nearby lcs structure has J-holomorphic elliptic curves for J-admissible

3) Apply the following theorem

Thm2 If α is rational then every non-constant J-curve

$u : \Sigma \to M$

closed Riemann surface

contains a Reeb curve.

(If J is admissible.)

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proof is via

Lemma: Let (M, λ, α, J) be an exact ISC structure with J admissible. Then

$u : \Sigma \to M$ is J-holomorphic

$\Rightarrow$ image $du(z) \subset V(u(z))$

$\forall z \in \Sigma$.

Proof:

$$0 \underset{\text{Stokes}}{\overset{\uparrow}{=}} \int_\Sigma u^* d\lambda = \underbrace{\cancel{\int_\Sigma d^{v} u \, d\lambda}}_{0} + \underbrace{\int_\Sigma d^\xi u \, d\lambda}_{>0}$$

If for some p $d^\xi u(p) \neq 0$

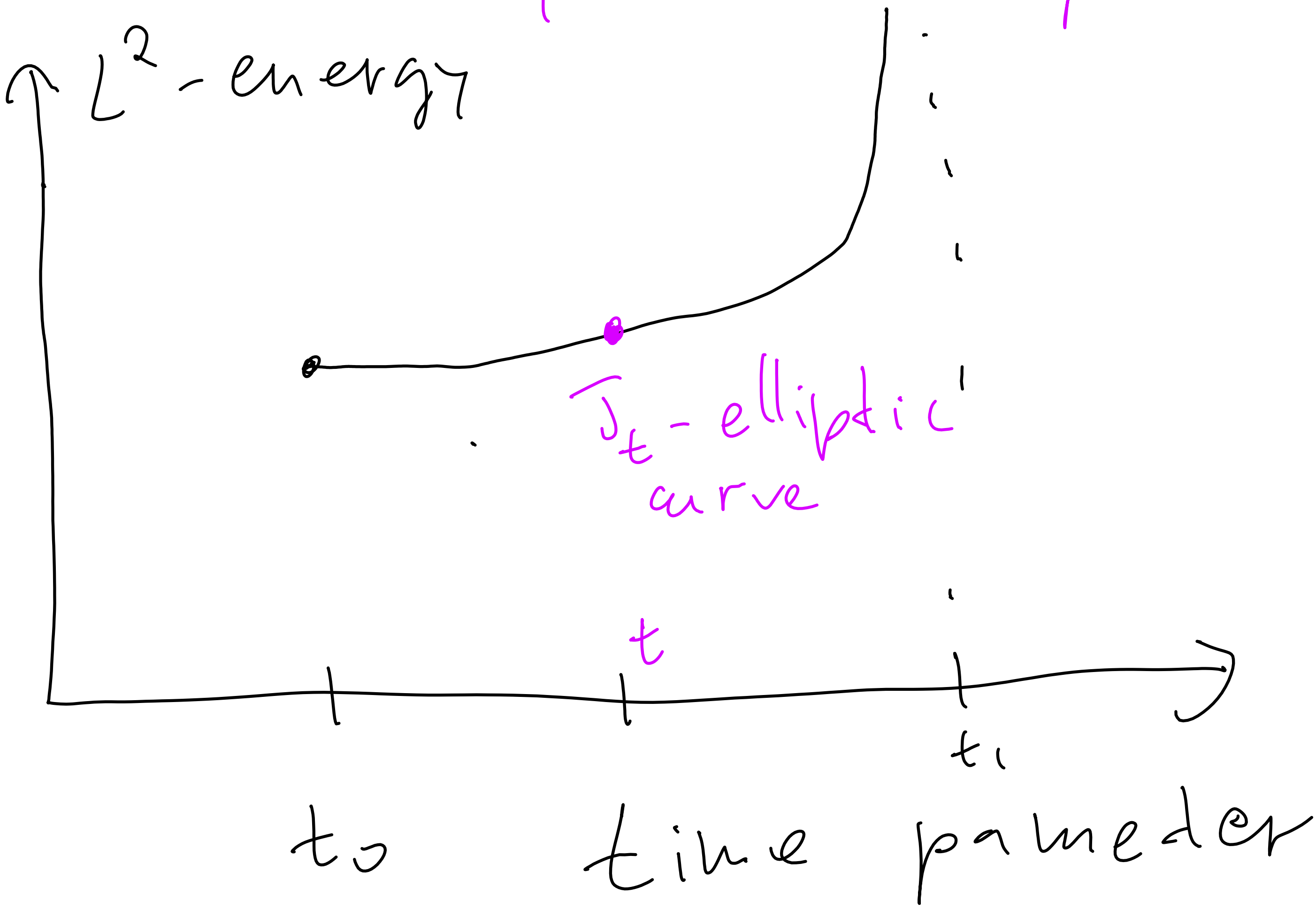
since J preserves ξ and

u is J-holomorphic.

$\Rightarrow \forall p \; d^\xi u(p) = 0.$ $\square$

why is theorem 1 only a local result?

no energy bounds means that we may have a phenomenon called sky catastrophe.



Open problem if this can exist.

An elementary version of the problem.

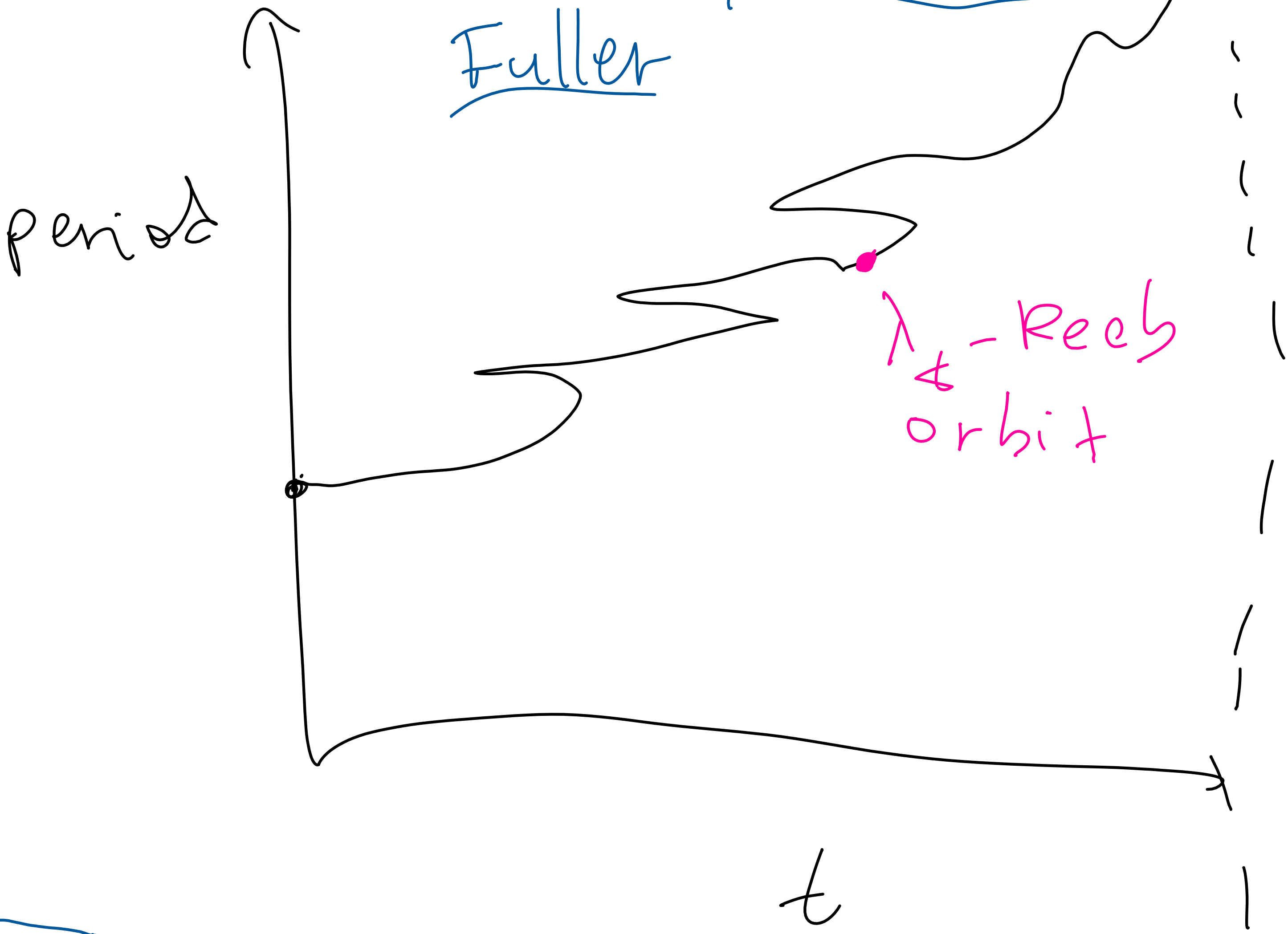
$M = S^3 \times S^1$, ω lcs-fication of the standard contact form λ on S^3.

$\{\lambda_t\}$, $t \in [0,1]$ a deformation, through contact forms.

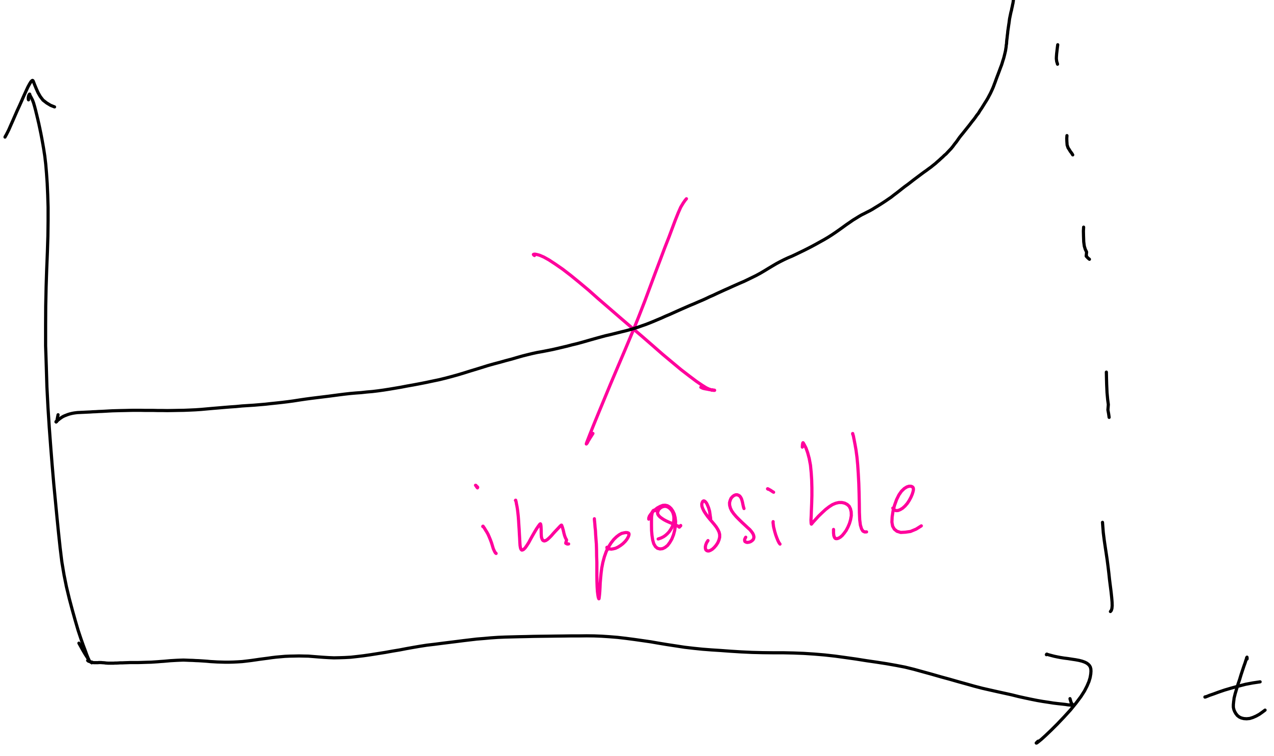
Can we find $\{\lambda_t\}$ so that there is a continuous family $s \mapsto o_s$, $\forall s$ o_s a $\{\lambda_t\}$-Reeb orbit and so that period $o_s \to \infty$
$s \to \infty$

blue sky catastrophy

Fuller



A bit of variational calculus.



In other words if a blue sky catastrophy $\{O_t\}$ exists then

$$\lim_{s \to \infty} \text{length } \pi^t(O_s) = \infty$$

You have to zig-zag.

Q: Does a Reeb orbit sky catastrophe exist?

Conjecture: Reeb orbit sky catastrophes are not C^0 stable.

Lcs homology (with Oh)

Another approach to CSW is via lcs-homology

For closed, exact lcs manifolds M.

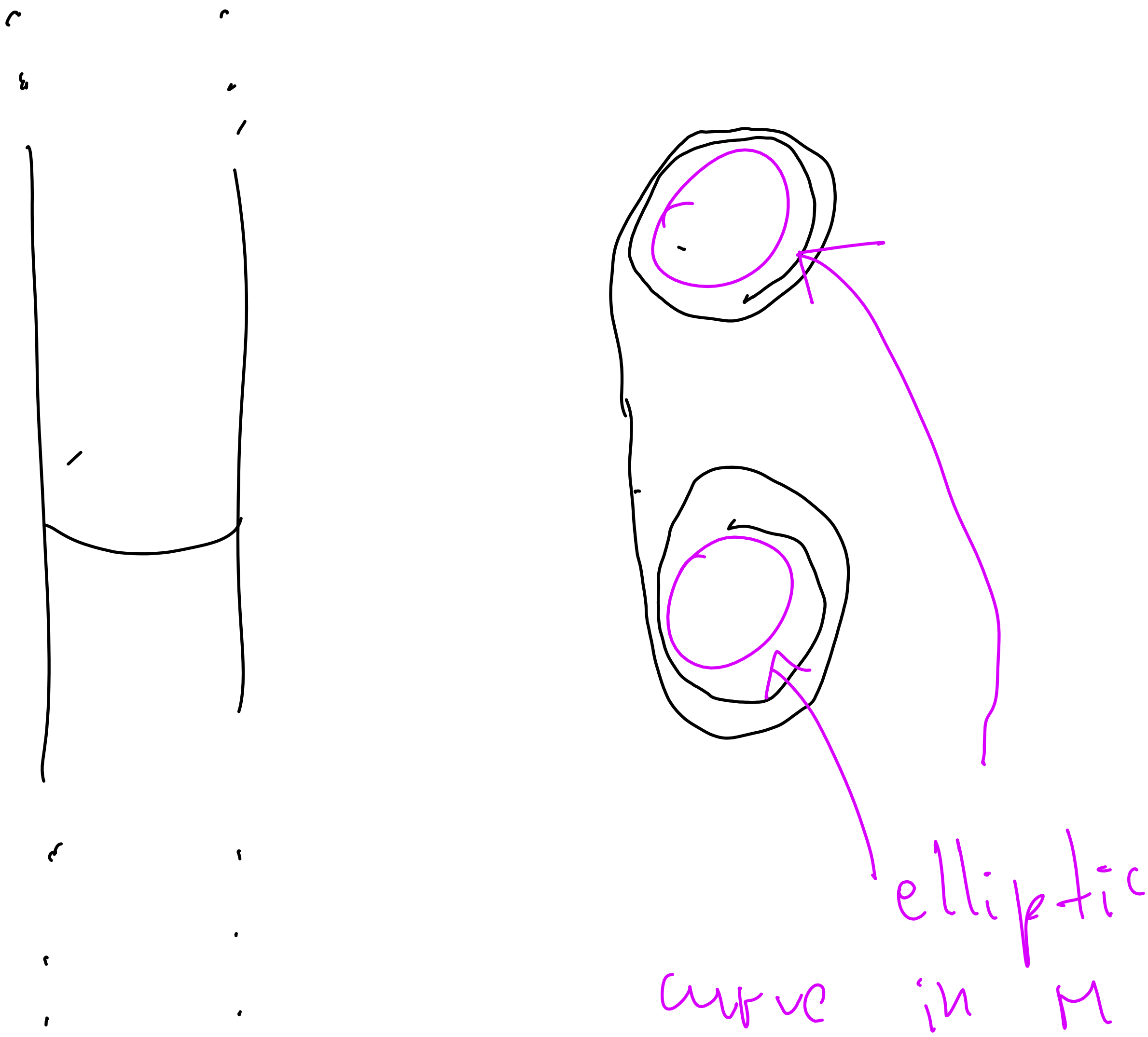
What are the generators?

One idea: they are elliptic curves $u: T^2 \to M$ for admissible J.

What are the instantons?

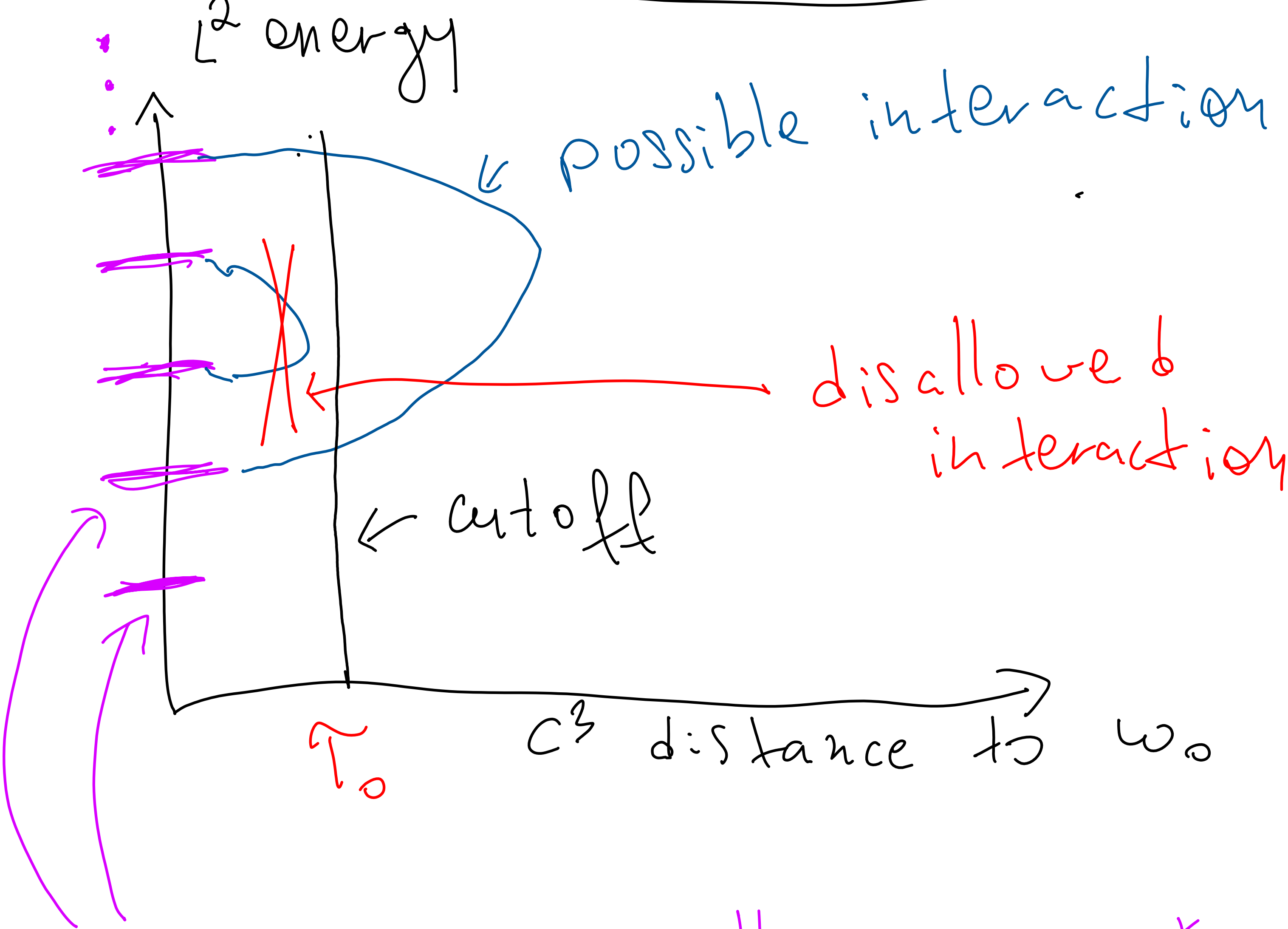
Finite energy holomorphic cylinders $\mathbb{R}\times S^1 \to M$ (as usual)

Finite energy forces ends of the cylinder to wrap around elliptic curves



The meaning and computation of GW invariant in Thm 1

$$\boxed{\begin{array}{c} M = S^{2k+1} \times S^1, \quad \omega_0 = d^\alpha \lambda \\ \lambda \text{ - standard contact form on } S^{2k+1} \end{array}}$$



components of $\mathcal{M}^{ell} = \coprod_{j \in \mathbb{N}} \mathbb{CP}^k$

Say we found a cutoff T_0 which is independent of the choice of deformation.

Then $\#_{reg} \mathcal{M}^{ell}$ makes sense as an invariant, in a T_0-neighborhood of ω_0, formal sum:

$$GW := \sum_{n \in \mathbb{N}} \#_{reg} (\mathcal{M}^{ell})_n \in \mathbb{Q}$$

$(\mathcal{M}^{ell})_n \cong \mathbb{C}P^k$ n'th component

Need to compute

$\#_{reg}(\mathcal{M}^{ell})_n$

This is done by relating this count to the classical Fuller index in dynamical systems.

Key ideas:

1) orientation of a Reeb torus u_0 is:

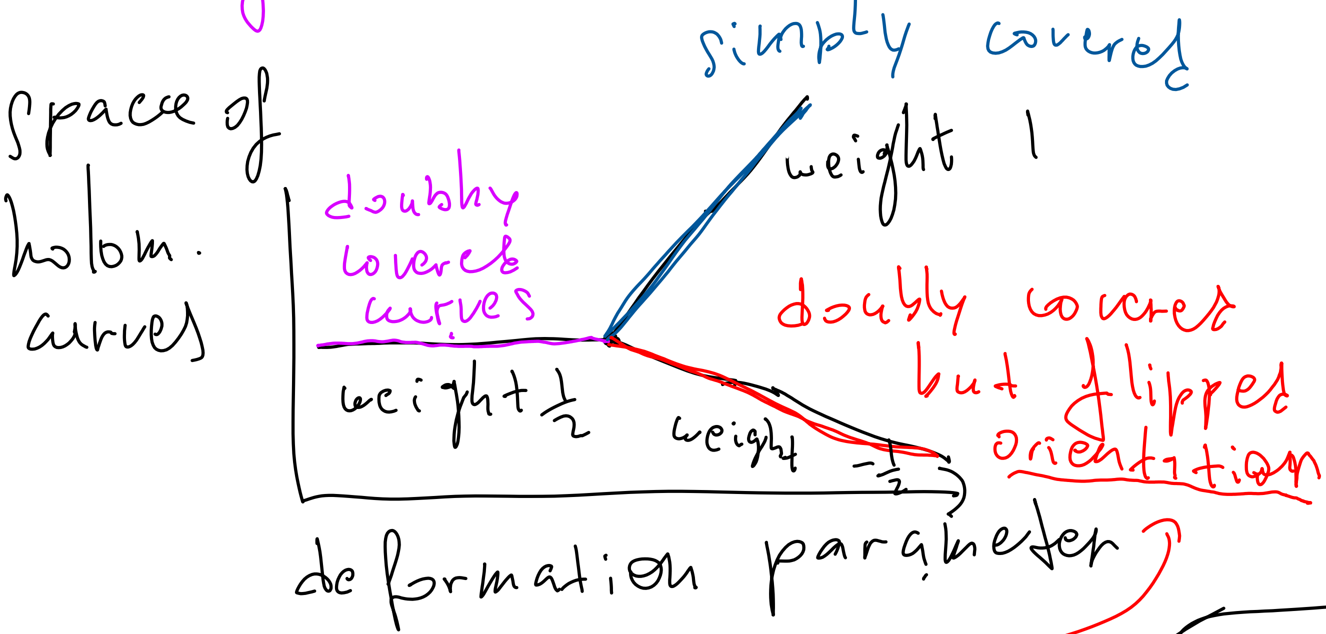
$$\boxed{(-1)^{CZ(0) + \text{normalization}}}$$

Conley-Zehnder index.

2) If o is non-degenerate as a Reeb orbit then u_0 is a <u>regular</u> curve.

(The associated CR operator is surjective)

3) Still need virtual moduli cycle because of phenomena like the period doubling bifurcation.



determinant line bundle orientation.

this is C^∞ stable