Locally conformally Kähler metrics. An overview.

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Locally Conformal Symplectic Manifolds:
Interactions and Applications

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LCK structures. Definition I

(M, I, g) Hermitian manifold, $\dim_{\mathbb{C}} M = n > 1$, ($I^2 = -1$, integrable),
$\omega(x, y) = g(Ix, y)$.

$$d\omega = \theta \wedge \omega, \qquad d\theta = 0$$

(θ is called *Lee form*, after H.-C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. J. Math. **65**, (1943), 433–438.)
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Complex submanifolds in LCK are LCK.

Characterization in terms of currents

(M, I, θ) complex manifold, $\dim_{\mathbb{C}} M \geq 2$, equipped with a closed 1-form.

Then M admits an LCK metric with Lee form θ if and only if there are no non-trivial positive currents which are (1, 1) components of d_θ-boundaries (here $d_\theta = d - \theta\wedge$). (Otiman)

Open question: LCS *versus* LCK

Find compact LCS manifolds which do not admit LCK structure.

Solved only in real dimension 4 using the classification of compact complex surfaces (Bande-Kotschik, Marrero & collaborators).

LCK structures. Definition II

Let (M, I) be a complex manifold covered by an atlkas $\{U_\alpha, \varphi_\alpha\}$ endowed with Kähler forms ω_α, s.t. the transition functions $\varphi_\alpha\varphi_\beta^{-1}$ are homotheties with respect to ω_β.

An LCK form on $(M, \{U_\alpha, \omega_\alpha\})$ is a Hermitian form ω which is conformally equivalent with each ω_α.

LCK structures. Definition III

(M, I) such that its universal cover $\pi : \tilde{M} \to M$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform group Γ acts on $(\tilde{M}, \tilde{\omega})$ by Kähler homotheties.

Definitions I-III appear in Vaisman's papers, starting with 1976.

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The *homothety character* is $\chi : \Gamma :\to \mathbb{R}^{>0}$, $\chi(\gamma) = \frac{\gamma^* \tilde{\omega}}{\tilde{\omega}}$.

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The minimal cover of an LCK manifold corresponds to a Γ on which χ is injective (Γ does not contain $\tilde{\omega}$-isometries).

The rank of $\mathrm{Im}(\chi)$ is the *LCK rank of* (M, I, ω).

LCK structures. The weight bundle

Let L be the local system corresponding to the character χ.

Then θ is a flat connection form in L and $\mathrm{Im}(\chi)$ its monodromy.

Call $\alpha \in \Lambda^* \tilde{M}$ *automorphic* if $\gamma^* \alpha = \chi(\gamma)\alpha$.

Automorphic forms on $\tilde{M}$ are identified with L-valued forms on M.

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The Morse-Novikov (twisted) cohomology of (M, ω, θ) is the cohomology of the complex $(\Lambda^* M, d_\theta := d - \theta\wedge)$.
It corresponds to the cohomology $H^*(M, L)$ of the local system L and is finite dimensional.

Examples

Almost all (known) non-Kähler compact complex surfaces (Vaisman, Gauduchon-O, Belgun, Brunella).

Particular examples and results on LCK surfaces: Apostolov, Dloussky, Fujiki, Gauduchon, Otiman, Pontecorvo...

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Some "toric Kato manifolds", generalization in higher dimensions of Kato surfaces, i.e. surfaces with global spherical shell (Istrati, Otiman, Pontecorvo, Ruggiero).

Kähler *versus* LCK

K	**LCK**
Blow up at points preserves the class	Yes (Tricerri, Vuletescu)
Blow up along submanifolds preserves the class	No (Yes, if and only if the submanifold has induced K structure, Verbitsky-Vuletescu-O)
Stability at small deformations	No (Inoue surfaces, Belgun). Yes for some particular subclass (LCK with potential, Verbitsky-O)
Killing fields are holomorphic on compact mfds	Yes, on compact mfds which are neither Hopf, nor have hyperkähler universal cover (Moroianu-Pilca)
Even odd betti numbers	No. There are examples with all b_k even (in $\dim_{\mathbb{C}} = 3$, by Oeljeklaus-Toma)
	Compact LCK manifolds cannot be Einstein (Madani-Moroianu-Pilca)

An LCK metric on a compact K manifold is automatically GCK (Vaisman) (proven for LCK spaces with singularities by Preda-Stanciu)

Vaisman manifolds: definition

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The condition is not conformally invariant. A Vaisman metric is Gauduchon ($d^*\theta = 0$).
On compact manifolds, a Vaisman metric, if it exists, is unique up to homothety in its conformal class.

Vaisman manifold: Examples

Diagonal Hopf manifolds $(\mathbb{C}^n \setminus 0)/\langle A \rangle$, $A \in \mathrm{GL}(n, \mathbb{C})$ diagonalizable, with eigenvalues of absolute value > 1;

All compact complex submanifolds of a Vaisman manifold are Vaisman;

Non-Kähler elliptic surfaces;

Some (but not all) small deformations of a compact Vaisman mfd are of Vaisman type.

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Non-Vaisman: Non-diagonal Hopf manifolds, Inoue surfaces, Kato manifolds, Oeljeklaus-Toma manifolds, blow-ups of LCK.

Vaisman manifolds: the canonical foliation

$\theta^\sharp$ and $I\theta^\sharp$ are commuting, Killing and real holomorphic vector fields.

Let $\Sigma := \langle \theta^\sharp, I\theta^\sharp \rangle$ be the foliation they generate. It is Riemannian and totally geodesic.
Regular: the leaf space is a manifold (projective).
Quasi-regular: compact leaves. The leaf space is an orbifold (projective).

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One has $d^c\theta = \omega - \theta \wedge I\theta$. Moreover, $\Sigma = \mathrm{Ker}(d^c\theta)$ and $d^c\theta$ is positive definite on $\Sigma^\perp$.

Characterization in terms of holomorphic flow

Let (M, ω, θ) be an LCK manifold equipped with a holomorphic and conformal $\mathbb{C}$-action without fixed points, which lifts to non-isometric homotheties on the Kähler cover $\tilde{M}$. Then (M, ω, θ) is conformally equivalent with a Vaisman manifold. (Kamishima-O)

A structure theorem

A compact Vaisman manifold of LCK rank 1 is biholomorphic isometric to a complex manifold obtained by the following receipe:

- Take (S, g_S, η) a compact Sasakian manifold;
- Let $(C(S) := S \times \mathbb{R}^{>0}, g := dt \otimes dt + t^2 g_S)$ be its Kähler cone;
- Let q be a non-trivial holomorphic homothety of $C(S)$ (along the generators).
- Then the compact complex manifold $M = C(S)/\langle q \rangle$ is Vaisman.

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Topology of compact Vaisman mfds: b_1 is odd, $H^*(M, L) = 0$ (de Leon *et al.* for LCS admitting a metric for which the Lee form is parallel.)

A structure theorem for q-r Vaisman

There exists a negative holomorphic orbifold line bundle L over X, such that M is biholomorphic to a $\mathbb{Z}$-quotient of the space $\mathrm{Tot}^\circ(L)$ of non-zero vectors in L.
The leaves of the canonical foliation are compact, and their preimages in $\mathrm{Tot}^\circ(L)$ coincide with the fibers of L.

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Not restrictive since: Any compact Vaisman manifold (M, I) admits a complex deformation (M, I') which is Vaisman and quasi-regular. Moreover, I' can be chosen arbitrarily close to I. (Verbitsky-O)

Sufficient conditions for compact LCK to be of Vaisman type

Einstein-Weyl LCK metrics are Vaisman. (Gauduchon)

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Toric LCK are Vaisman (Istrati). Toric Vaisman have $b_1 = 1$ and kod $= -\infty$ (Madani-Moroianu-Pilca).

LCK manifolds with potential. Definition

A Kähler cover $\Gamma \longrightarrow (\tilde{M}, \tilde{\omega}) \xrightarrow{\pi} (M, \omega, \theta)$ admits strictly positive and automorphic global potential:

$$\tilde{\omega} = dd^c\varphi, \qquad \gamma^*\varphi = \chi(\gamma)\varphi$$

In this case $\pi^*\theta = d\log\varphi$ and $\pi^*\omega = \frac{dd^c\varphi}{\varphi}$.

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Equivalent definitions on (M):

1. $\omega = d_\theta d^c_\theta \varphi_0$, where $\varphi_0 : M \longrightarrow \mathbb{R}^{>0}$.
2. $d^c\theta = \omega - \theta \wedge I\theta$

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If M compact and (M, I, ω) is LCK with potential, then any small deformation (M, I_t) admits LCK metrics with potential. (Verbitsky-O)

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Compact LCK not admitting LCK potential: Inoue surfaces (Otiman), Oeljeklaus-Toma manifolds (Kasuya, Otiman).

The Kähler cover of LCK manifolds with proper potential

An LCK potential is proper if and only if $\Gamma \cong \mathbb{Z}$ (i.e. the LCK rank is 1).
The $\mathbb{Z}$-cover of an LCK manifold with proper potential, $\dim_{\mathbb{C}} \geq 3$ can be completed with only 1 point to a Stein variety (in general non-smooth).

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The restriction on dimension: we use a theorem of Rossi-Andreotti, Siu: Let S be a compact strictly pseudoconvex CR-manifold, $\dim_{\mathbb{R}} S \geq 5$, and let $H^0(\mathcal{O}_S)_b$ the ring of bounded CR-holomorphic functions. Then S is the boundary of a Stein variety M with isolated singularities, such that $H^0(\mathcal{O}_S)_b = H^0(\mathcal{O}_M)_b$, where $H^0(\mathcal{O}_M)_b$ denotes the ring of bounded holomorphic functions. Moreover, M is defined uniquely: $M = \mathrm{Spec}(H^0(\mathcal{O}_S)_b)$.

The Kähler cover of LCK manifolds with proper potential

The $\mathbb{Z}$ cover has the structure of a *closed algebraic cone*, *id est* an affine variety $\mathcal{C}$ admitting a $\mathbb{C}^*$-action ρ with a unique fixed point x_0, called *the origin*, and satisfying the following:

- $\mathcal{C}$ is smooth outside of x_0,
- ρ acts on the Zariski tangent space $T_{x_0}\mathcal{C}$ with all eigenvalues $|\alpha_i| < 1$.

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A compact LCK manifold with potential (M, I, ω, θ) can be deformed to (M, I, ω', θ') with proper potential. (Verbitsky-O)

Embedding LCK manifold with proper potential into Hopf manifolds

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(M, I), $\dim_{\mathbb{C}} M \geq 3$, is of Vaisman type if and only if it can be holomorphically embedded in a diagonal Hopf manifold.

A compact Sasakian manifold admits a CR embedding into a diffeomorphism sphere, preserving the Reeb fields (Verbitsky-O).

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Extension to LCS of type I: David Martinez Torres & collaborators.

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Compact LCK with potential, $\dim_{\mathbb{C}} \geq 3$, can be deformed to Vaisman manifolds. In particular, they have the same topology as Vaisman manifolds.

A criterion for the existence of LCK metrics with potential metrics

(M, I, ω, θ) compact, admits a holomorphic S^1 action which lifts to an action by homotheties (and not only isometries) of the Kähler cover. (Verbitsky-O)

The converse is also true: use embedding in Hopf and logarithm of the monodromy.

The set of Lee classes

For (M, I) of LCK type, let

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The same holds for LCK with potential of $\dim_{\mathbb{C}} \geq 3$ (Verbitsky-O).
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Follows from:

> Let (M, θ, ω) be a compact LCK manifold with potential, and $H^{1,0}(M)$ denote the space of holomorphic 1-forms on M. Then $H^1(M, \mathbb{C}) = H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \oplus \langle\theta\rangle$.
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Sharp contrast with Inoue surfaces where $\mathcal{L}$ is a single point.
(Apostolov-Dloussky, Otiman)