Level structures and images of the Steinberg module for surfaces with marked points

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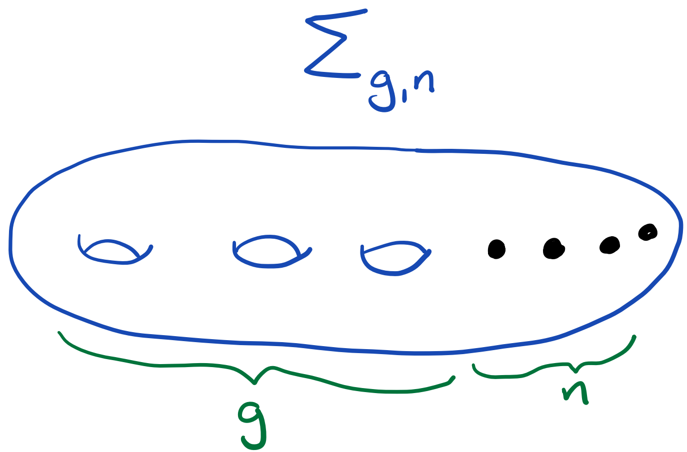
Surface of genus g with n marked points

- Surface $\Sigma_{g,n}$ of genus g with finite marked point set $P \subset \Sigma_{g,n}$ with $|P| = n$.
- If $n = 0$ we write Σ_g.



Moduli Space and Teichmüller space

- **Moduli Space** is

  $$\mathscr{M}_{g,n} \equiv \{ \text{ Complete fixed-area hyp. structures on } \Sigma_{g,n} - P \ \}$$

- $\mathscr{M}_{g,n}$ is orbifold with $\mathbf{R}$-dimension $6g - 6 + 2n$ and orbifold fundamental group $\mathrm{Mod}_{g,n}$.
- Universal orbifold cover of $\mathscr{M}_{g,n}$ is **Teichmüller space** $\mathscr{T}_{g,n}$ (topologically $\mathbf{R}^{6g-6+2n}$ so contractible).

(Pure) mapping class group

Definition

(Pure) mapping class group of $\Sigma_{g,n}$ is

$$\mathrm{Mod}_{g,n} \equiv \mathrm{Diff}^{+}(\Sigma_{g,n}, P) \Big/ \mathrm{Diff}^{0}(\Sigma_{g,n}, P).$$

- Marked points fixed **pointwise**.
- Index $n!$ in usual $\mathrm{Mod}_{g,n}$
- Today all mapping classes are pure.

Why study cohomology of $\mathrm{Mod}_{g,n}$?

- Nice to be at a conference where no justification necessary!
- But also, $\mathscr{M}_{g,n}$ is $B\,\mathrm{Diff}(\Sigma_{g,n})$ so cohomology gives characteristic classes of surface bundles.
- $$\mathrm{H}^*(\mathrm{Mod}_{g,n};\mathbf{Q}) \cong \mathrm{H}^*(\mathscr{M}_{g,n};\mathbf{Q})$$

vcd of mapping class group

Theorem (Harer)

$\mathrm{Mod}_{g,n}$ *has* **virtual cohomological dimension**

$$\nu = \mathrm{vcd}(\mathrm{Mod}_{g,n}) = \begin{cases} 4g+n-4, & g \geq 1 \text{ and } n \geq 1 \\ 4g-5, & g \geq 1 \text{ and } n = 0 \\ n-3, & g = 0 \text{ and } n \geq 3 \end{cases}$$

Stable cohomology of $\mathscr{M}_{g,n}$

- Madsen-Weiss give graded ring isomorphism in degrees below $\frac{2}{3}(g-1)$

$$\mathbf{Q}[u_1, \cdots, u_n, \kappa_1, \kappa_2, \cdots] \to \mathsf{H}^*(\mathscr{M}_{g,n}; \mathbf{Q})$$

- Miller Mumford Morita classes $\kappa_i \in \mathsf{H}^{2i}(\mathscr{M}_{g,n}; \mathbf{Q})$
- Euler classes of tangent directions moving jth marked point $u_j \in \mathsf{H}^2(\mathscr{M}_{g,n}; \mathbf{Q})$

Unstable cohomology exists

- If cohomology for fixed topology were mostly stable classes then Euler characteristic would be at most polynomial in g and n. Harer-Zagier and Bini-Harer give super-exponential in both.
- **Lots** of unstable cohomology.

Unstable results

Unstable cohomology only possible in higher degrees

Theorem (Church-Farb-Putman, Morita-Sakasai-Suzuki)

For $g \geq 2$

$$\mathrm{H}^{4g-5}(\mathrm{Mod}_g; \mathbf{Q}) = 0$$

Theorem (Chan-Galatius-Payne)

For $g \geq 7$

$$\mathrm{H}^{4g-6}(\mathrm{Mod}_g; \mathbf{Q}) \neq 0$$

Level-ℓ subgroup

Definition

$\ell \geq 2$. For P marked point set of $\Sigma_{g,n}$ **level-ℓ subgroup** $\mathrm{Mod}_{g,n}[\ell]$ of $\mathrm{Mod}_{g,n}$ is kernel of action on $\mathrm{H}_1(\Sigma_{g,n}, P; \mathbf{Z}/\ell)$

- Have corresponding **moduli space for surfaces with level-ℓ structure** $\mathscr{M}_{g,n}[\ell]$
- Cohomology of $\mathscr{M}_{g,n}[\ell]$ gives characteristic classes for surface bundles with level-ℓ structure.

Top cohomology virtually nontrivial for Mod_g

Theorem (Fullarton-Putman)

For $g \geq 1$ and p prime dividing ℓ

$$\dim_{\mathbf{Q}} \mathrm{H}^{4g-5}(\mathrm{Mod}_g[\ell]; \mathbf{Q}) \geq \frac{1}{g} p^{2g-1} \prod_{k=1}^{g-1} (p^{2k} - 1) p^{2k-1}$$

Main Theorem

Theorem (Brendle-B-Putman)

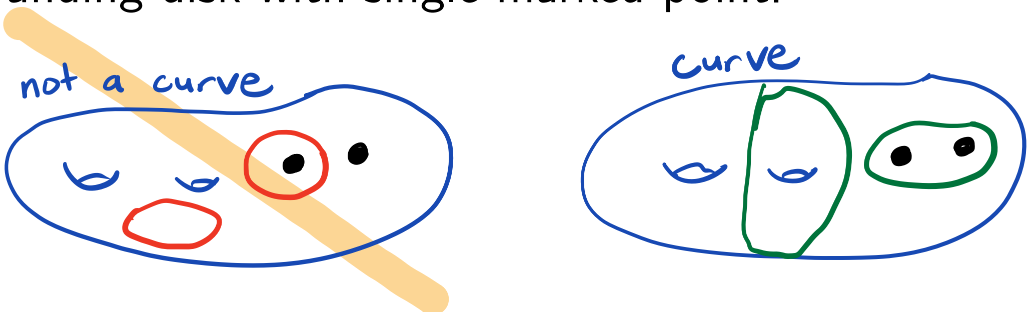
For $g \geq 1$, $n \geq 2$ and $\ell \geq 2$

$$\dim_{\mathbf{Q}} \mathrm{H}^{\nu}(\mathrm{Mod}_{g,n}[\ell]; \mathbf{Q}) \geq \left(\prod_{k=1}^{g-1} (k\ell^{2g} - 1) \right) \cdot \underbrace{\dim_{\mathbf{Q}} \mathrm{H}^{4g-3}(\mathrm{Mod}_{g,1}[\ell]; \mathbf{Q})}_{\text{Fullerton–Putman}}$$

n

Curves

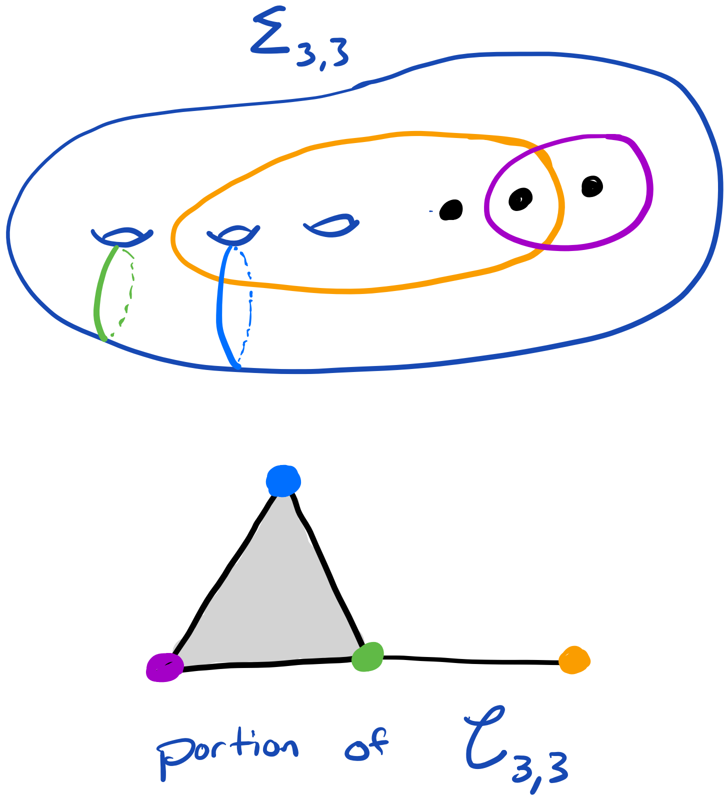
- A **curve** is an isotopy class of embedded circle in $\Sigma_{g,n}$ not bounding disk with single marked point.



- Fix hyp. metric on $\Sigma_{g,n}$ then curves have unique geodesic reps with minimal intersection.
- **Curve system** is disjoint set of curves.

Curve complex

- **Curve complex** $\mathscr{C}_{g,n}$ is flag complex with vertex set curves in $\Sigma_{g,n}$ and simplex for each curve system



Homotopy type of curve complex

Theorem (Harer)

Curve complex has homotopy type of infinite wedge sum of spheres of dimension

$$\lambda = \begin{cases} 2g+n-3, & g \geq 1 \text{ and } n \geq 1 \\ 2g-2, & g \geq 1 \text{ and } n = 0 \\ n-4, & g = 0 \text{ and } n \geq 4 \end{cases}.$$

Harer Bordification

- Harer bordifies $\mathscr{T}_{g,n}$ to get $\overline{\mathscr{T}}_{g,n}$ with boundary

$$\partial\overline{\mathscr{T}}_{g,n} = \overline{\mathscr{T}}_{g,n} - \mathscr{T}_{g,n}$$
$$\simeq \mathscr{C}_{g,n}$$
$$\simeq \vee^{\infty} S^{\lambda}$$

- Harer work mirrors Borel-Serre bordification of symmetric space

Mapping class group is virtual duality group

Theorem (Harer)

$\mathrm{Mod}_{g,n}$ *is a* **virtual duality group** *with*

$$\mathrm{vcd}(\mathrm{Mod}_{g,n}) = \begin{cases} 4g+n-4, & g \geq 1 \text{ and } n \geq 1 \\ 4g-5, & g \geq 1 \text{ and } n = 0 \\ n-3, & g = 0 \text{ and } n \geq 3 \end{cases}$$

and dualizing module the **Steinberg module**

$$\mathrm{St}_{g,n} = \widetilde{\mathrm{H}}_{\lambda}(\mathscr{C}_{g,n})$$

Steinberg module facts

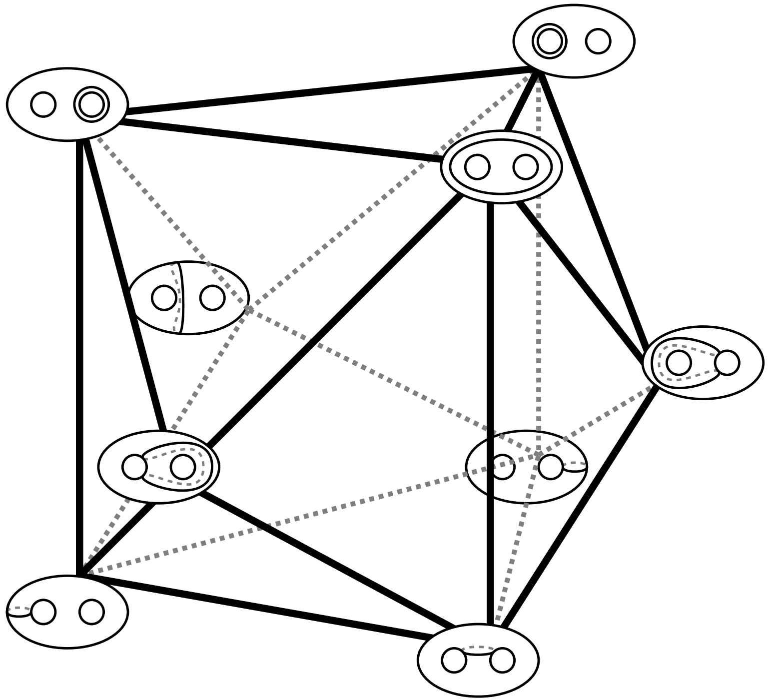
Theorem (B, Birman-B-Menasco)

$\mathrm{St}_{g,1}$ *and* $\mathrm{St}_{0,n}$ *have finite virtually free* Mod-*module resolutions with last two terms giving finite* Mod-*module presentations .*

Theorem (B, Birman-B-Menasco)

$\mathrm{St}_{g,1}$ *and* $\mathrm{St}_{0,n}$ *are cyclic* Mod-*modules hence singleton generator is nontrivial sphere.*

St_2 generator



Top cohomology and Steinberg coinvariants

- Now return to the rational Steinberg module

$$\mathsf{St}_{g,n} = \widetilde{\mathsf{H}}_{\lambda}(\mathscr{C}_{g,n}; \mathbf{Q})$$

- By duality if $\Gamma < \mathsf{Mod}_{g.n}$ finite index then

$$\mathsf{H}^{\nu}(\Gamma; \mathbf{Q}) \cong \mathsf{H}_0(\Gamma; \mathsf{St}) \cong (\mathsf{St})_{\Gamma}$$

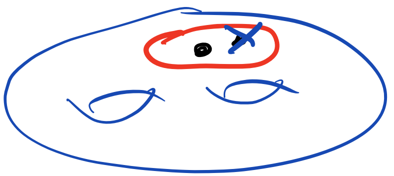
- Hence cohomology in vcd if $\mathsf{Mod}_{g,n}[\ell]$ in vcd is coinvariants

$$(\mathsf{St})_{\mathsf{Mod}_{g,n}[\ell]}$$

Forgetful map on $\mathscr{C}_{g,n}$

- For $n \geq 2$ have **partially defined** "forgetful map"

$$\mathscr{C}_{g,n} \to \mathscr{C}_{g,n-1}$$



which "forgets" nth marked point.

- Let $\mathbf{A}_{g,n}$ be set of curves in $\Sigma_{g,n}$ bounding dsk containing nth marked point and one other marked point.

- Forgetful map defined **except** on $\mathbf{A}_{g,n}$

Inductive description of homotopy type of curve complex

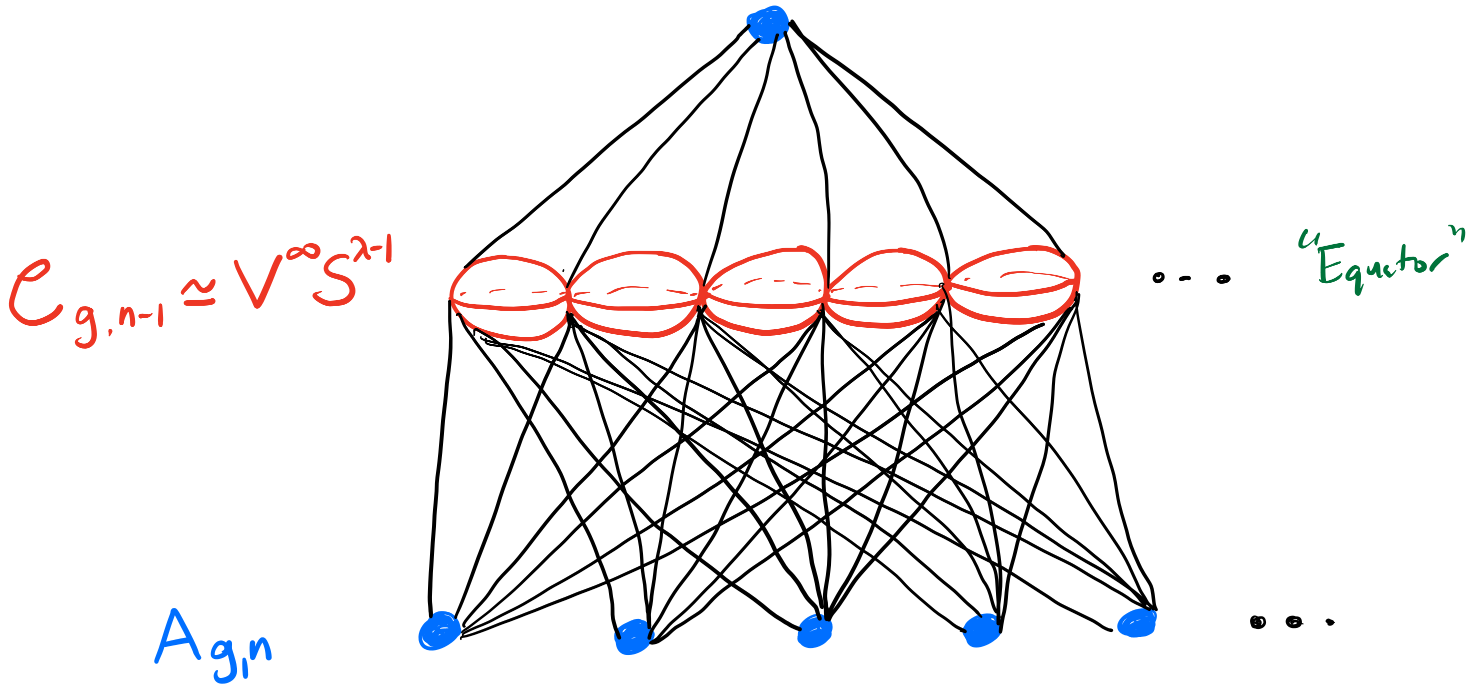
- For $g \geq 1$ and $n \geq 2$ such that $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$ there is a $\mathrm{Mod}_{g,n}$-equivariant homotopy equivalence

discrete set

$$\mathscr{C}_{g,n} \simeq \mathbf{A}_{g,n} * \mathscr{C}_{g,n-1}$$

We give new proof using "Hatcher Flow".

Homotopy picture of curve complex



$\mathbf{X}_{g,n}$ homotopy equivalent to $\mathscr{C}_{g,n-1}$

- Let $\mathbf{X}_{g,n} \subset \mathscr{C}_{g,n}$ be full subcomplex spanned by vertices of $\mathscr{C}_{g,n}$ not in $AC_{g,n}$

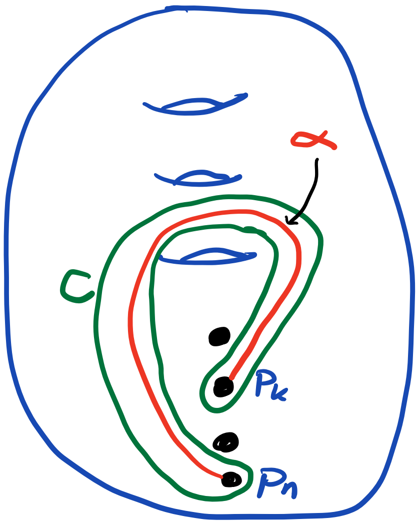
Proposition

For all $c \in \mathbf{A}_{g,n}$ we have deformation retraction

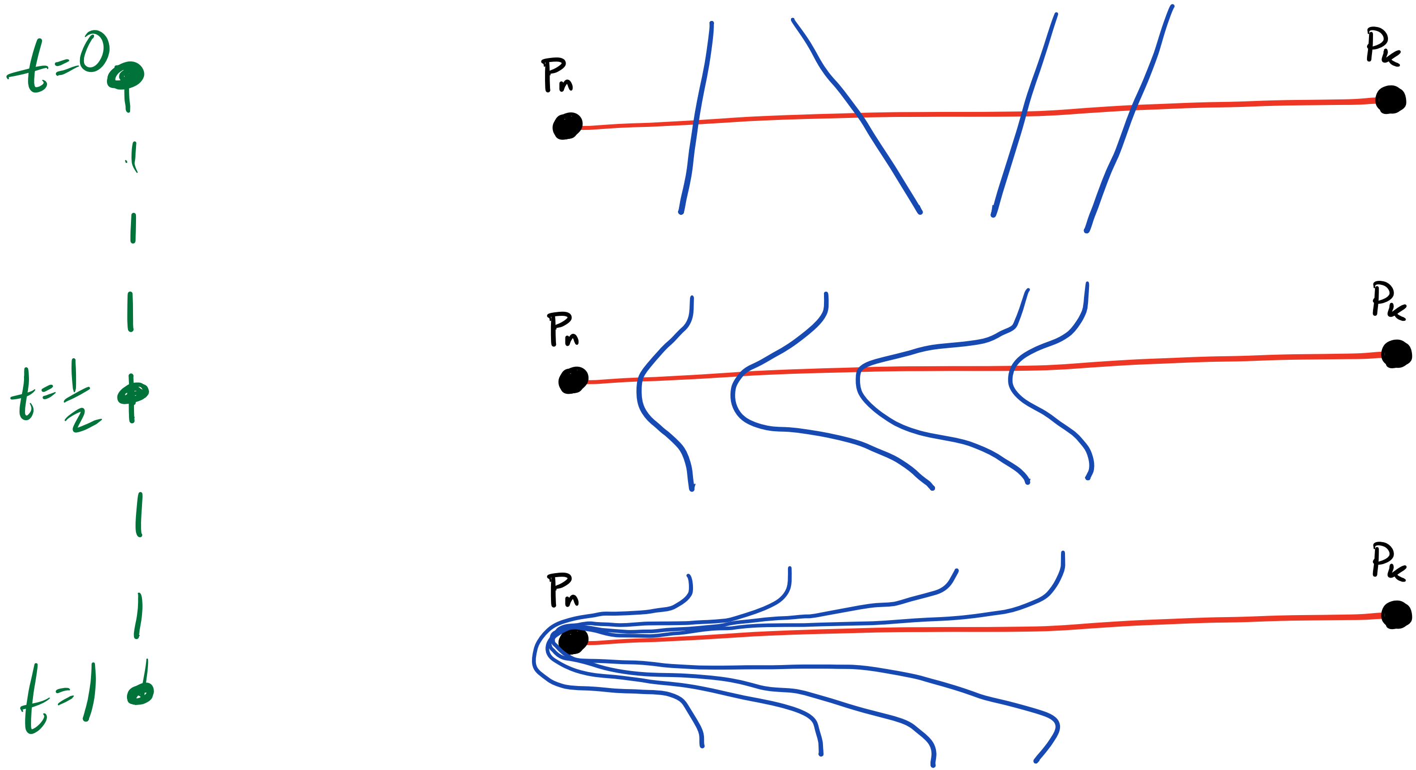
$$r_c : \mathbf{X}_{g,n} \to \mathrm{lk}_{\mathscr{C}_{g,n}}(c)$$

Proof.

Set α to be unique arc connecting nth marked point to other marked point disjoint from c. $\square$



Hatcher flow



Consequences for $\mathsf{St}_{g,n}$

- Joining wedge sum of spheres with discrete set gives wedge sum of spheres of increased dimension
- Hence

$$\mathsf{St}_{g,n} \cong \widetilde{\mathsf{H}}_{\lambda}(\mathscr{C}_{g,n}; \mathbf{Q}) \cong \widetilde{\mathsf{H}}_{0}(\mathbf{A}_{g,n}; \mathbf{Q}) \otimes \widetilde{\mathsf{H}}_{\lambda-1}(\mathscr{C}_{g,n-1}; \mathbf{Q})$$

Inductive Steinberg description

- For S a set let $\widetilde{\mathbf{Q}}[S]$ denote the kernel of augmentation

$$\varepsilon : \mathbf{Q}[S] \to \mathbf{Q}$$

- Get inductive description:

$$\mathrm{St}_{g,n} \cong \widetilde{\mathbf{Q}}[\mathbf{A}_{g,n}] \otimes \mathrm{St}_{g,n-1}$$

- Enough to find $\mathrm{Mod}_{g,n}[\ell]$-invariant quotients of factors on RHS.

Action of $\mathrm{Mod}_{g,n}[\ell]$ on $\mathbf{A}_{g,n}$

- Now view $\mathbf{A}_{g,n}$ as set of **arcs** (embedded intervals connecting distinct elts. of P)
- Let $\mathbf{A}_{g,n}^k \subset \mathbf{A}_{g,n}$ be arcs connecting kth and nth marked points p_k and p_n.

Lemma

Action of $\mathrm{Mod}_{g,n}[\ell]$ *on arc set* $\mathbf{A}_{g,n}^k$ *has* ℓ^{2g} *orbits.*

Source of lower bound formula

Proposition

Action of $\mathrm{Mod}_{g,n}[\ell]$ *on arc set* $\mathbf{A}^{\cancel{k}}_{g,n}$ *has* $(n-1)\ell^{2g}$ *orbits.*

- Adding nth marked point multiplies size of basis of coinvariants by $(n-1)\ell^{2g} - 1$
- Let $\mathbf{A}^{k}_{g,n} \subset \mathbf{A}_{g,n}$ be arcs connecting kth and nth marked points p_k and p_n.

Applications

For $g \geq 2$ and $n \geq 1$

Conjecture (Looijenga)

$$\mathrm{CohCD}(\mathscr{M}_g) \leq g - 2$$

Theorem (Fullarton-Putman)

$$\mathrm{CohCD}(\mathscr{M}_g) \geq g - 2$$

Theorem (Brendle-B-Putman)

$$\mathrm{CohCD}(\mathscr{M}_{g,n}) \geq g - 1$$

$$\mathrm{CohCD}(\mathscr{M}_{g,n}) \leq \mathrm{CohCD}(\mathscr{M}_g) + 1$$