Instability mechanisms for inverse problems

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Outline

1. Background
2. Entropy and capacity estimates
3. Examples

Image deblurring

Blurred versions of an image $f \in L^2(\mathbb{R}^2)$:

f

$\chi_1 * f$

$\chi_2 * f$

Instability

Try to recover an image f from its blurred version $f_b = \chi * f$. Fourier transform: $\hat{f}_b(\xi) = \hat{\chi}(\xi)\hat{f}(\xi)$. Different scenarios:

- $\hat{\chi} \in C_c^\infty(\mathbb{R}^2) \rightsquigarrow$ high frequencies are lost completely
- $\hat{\chi}(\xi) \sim e^{-c|\xi|} \rightsquigarrow$ high frequencies exponentially damped
- $\hat{\chi}(\xi) \sim |\xi|^{-s} \rightsquigarrow$ high frequencies polynomially damped

If one measures $m = \chi * f + \varepsilon$ where ε is noise, naive reconstruction (multiply by $\frac{1}{\hat{\chi}(\xi)}$ on Fourier side) gives

$$f_{\mathrm{naive}} = f + \mathscr{F}^{-1}\{\frac{1}{\hat{\chi}(\xi)}\hat{\varepsilon}(\xi)\}.$$

High frequency noise can lead to huge errors in reconstruction!

Instability

Heuristics:

- smoothing (blurring) implies instability
- strong smoothing (fast decay of singular values) implies strong instability

In deblurring, the forward operator was the simple linear operator

$$F : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad F(f) = \mathscr{F}^{-1}\{\hat{\chi}(\xi)\hat{f}(\xi)\}$$

and the analysis could be done directly on the Fourier side.

What happens for more complicated inverse problems?

Instability

We give a framework for studying rigorously the inherent instability in various linear and nonlinear inverse problems, e.g.

- (geodesic) X-ray/Radon transforms also with limited data
- analytic/unique continuation
- control and inverse problems for heat/wave equations
- Calderón problem

We identify three mechanisms for instability:

1. Strong global smoothing
2. Microlocal smoothing
3. Weak global smoothing ("iterated small regularity gain")

Abstract problem

Abstract inverse problem. Consider a map $F : X \to Y$ between metric spaces. Given $y \in Y$, find $x \in X$ with

$$F(x) = y.$$

Conditions for well-posedness [Hadamard 1902]:

1. (Existence) Given $y \in Y$, there is $x \in X$ with $F(x) = y$.
2. (Uniqueness) If $F(x_1) = F(x_2)$, then $x_1 = x_2$.
3. (Stability) The solution x depends continuously on y.

In IP stability typically fails, but may have conditional stability. Important for convergence guarantees for statistical algorithms [Abraham, Giordano, Monard, Nickl, Paternain, 2019–].

Conditional stability

Fact. If $F : X \to Y$ is an injective continuous map and $K \subset X$ is compact, then $F|_K$ is a homeomorphis.

Restricting to a compact set $K \subset X$ (a priori bounds) gives conditional stability: there is a *modulus of continuity* $\omega = \omega_{F,X,Y,K}$ so that

$$d_X(x_1, x_2) \leq \omega(d_Y(F(x_1), F(x_2))), \qquad x_j \in K.$$

Examples:

1. If $\omega(t) = t$, one has Lipschitz stability.
2. If $\omega(t) = t^\alpha$, one has Hölder stability.
3. If $\omega(t) = |\log t|^{-\sigma}$, one has logarithmic stability.

Calderón problem/EIT

Example. Consider $\mathrm{div}(\gamma \nabla u) = 0$ in $\Omega \subset \mathbb{R}^n$. In this case

$$
\begin{aligned}
X &= (L^\infty_+(\Omega), \|\cdot\|_{L^\infty}), \\
Y &= (B(H^{1/2}, H^{-1/2}), \|\cdot\|_* = \|\cdot\|_{H^{1/2} \to H^{-1/2}}), \\
F &: \quad \gamma \mapsto \Lambda_\gamma \quad \text{(Dirichlet-to-Neumann map)}.
\end{aligned}
$$

- Uniqueness is highly nontrivial [Sylvester-Uhlmann 1987, Astala-Päivärinta 2006, ...].
- Stability fails: F^{-1} is not continuous $F(X) \to X$ [Alessandrini 1988].

Conditional stability

Logarithmic stability [Alessandrini 1988]: if $n \geq 3$ and $K = \{\gamma \in L^\infty(\Omega) : \gamma \geq E^{-1}, \ \|\gamma\|_{H^{n/2+2+\varepsilon}(\Omega)} \leq E\}$, then

$$\|\gamma_1 - \gamma_2\|_{L^\infty} \leq \omega(\|F(\gamma_1) - F(\gamma_2)\|_*), \qquad \gamma_j \in K,$$

where ω is a logarithmic modulus of continuity.

Many improvements [Barceló-Faraco-Ruiz 2007, ...]. Also, if K is contained in an N-dim. space, get Lipschitz stability with constant blowing up as $N \to \infty$ [Alessandrini-Vessella 2005].

Exponential instability [Mandache 2001]: logarithmic stability is optimal, i.e. if the above estimate holds for some ω, then $\omega(t) \geq c|\log t|^{-\sigma}$ for some $\sigma = \sigma(n)$.

Instability

The argument in [Mandache 2001] was based on

- capacity estimates for $K \subset X$
- entropy estimates for $F(K) \subset Y$

These estimates were proved by ad hoc constructions, e.g. by spherical harmonics and separation of variables in the ball.

We will replace these ad hoc constructions by structural properties of the forward operator and related spaces.
This works for many inverse problems (not just EIT), and for general geometries and coefficients.

Message: while a stability result is “hard” (requires a quantitative uniqueness result), an instability result is “soft” (follows from structural “compression” properties).

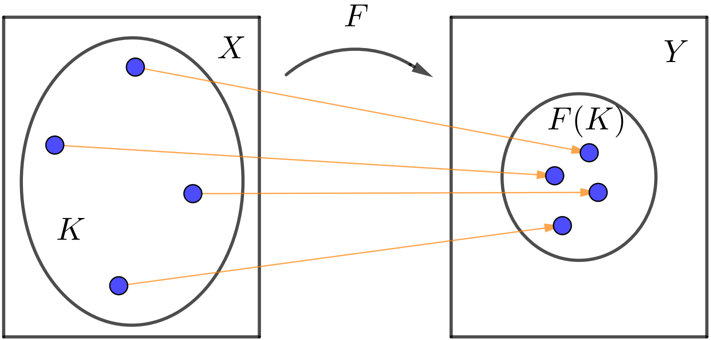
Outline

1. Background
2. Entropy and capacity estimates
3. Examples

Instability

Consider a map $F : X \to Y$ between metric spaces, and the inverse problem of solving $F(x) = y$ when $x \in K$.

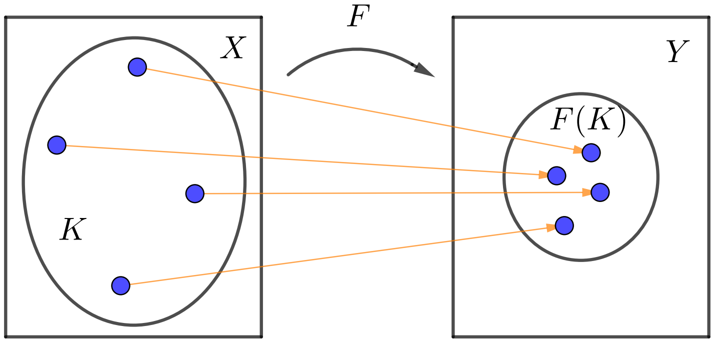
One expects instability if F "strongly compresses distances".



Instability [Mandache 2001]

Expect instability if K is "extended" ($\exists$ large ε-discrete sets), whereas $F(K)$ is "compressed" ($\exists$ relatively small δ-nets).

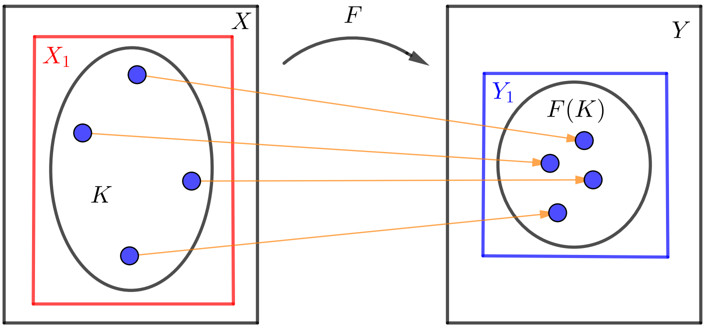
Then pigeonhole principle $\implies$ instability.



Instability

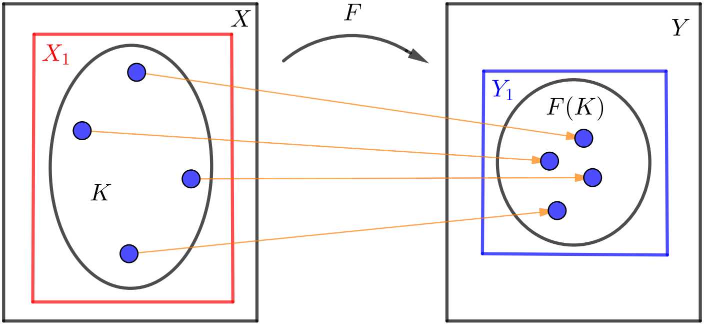
X and Y are often Banach spaces and K is a bounded set in some subspace $X_1 \subset X$. Suppose that:

$F(K) \subset Y_1$ where $Y_1 \subset Y$ is a "compressed" subspace.



Instability

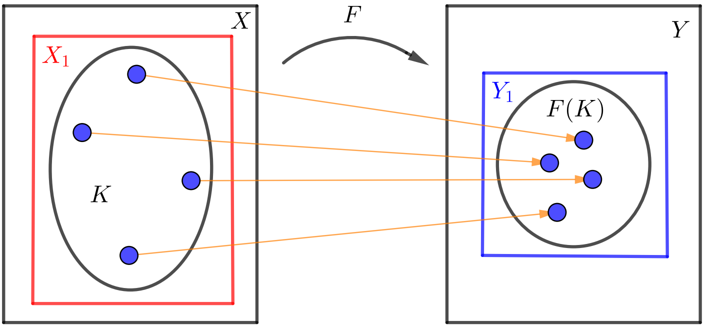
Then ε-discrete sets in K can be studied via the embedding $i : X_1 \to X$, and δ-nets of $F(K)$ are related to the embedding $j : Y_1 \to Y$.



Instability

If $F(K) \subset Y_1$, enough to study embeddings between function spaces (the forward operator "disappears"!). The ideal tool:

Capacity and entropy numbers (see [Edmunds-Triebel 2008]).



Entropy and capacity numbers

Let $A : X \to Y$ be a bounded operator between Banach spaces. Let $U_X = \{x \in X \,:\, \|x\|_X \le 1\}$. For $k \ge 1$, define

$$e_k(A) = \inf\{\delta \,:\, \text{there is a } \delta\text{-net of } A(U_X) \text{ with } 2^{k-1} \text{ elements}\},$$
$$c_k(A) = \sup\{\varepsilon \,:\, A(U_X) \text{ has an } \varepsilon\text{-discrete set of} > 2^{k-1} \text{ elements}\}.$$

Enough to consider entropy numbers, since $c_k(A) \sim e_k(A)$.
In our case, study $e_k(i : X_1 \to X)$ and $e_k(j : Y_1 \to Y)$.

The numbers $e_k(A)$ measure compactness of A:

- A is compact iff $e_k(A) \to 0$ as $k \to \infty$;
- A has finite rank iff $e_k(A)$ decay exponentially.
- In Hilbert spaces, $e_k(A)$ related to singular values $\sigma_k(A)$.

Entropy number bounds

Theorem. (Smooth spaces are compressed)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Then

$$e_k(i : H^{s+\delta}(\Omega) \to H^s(\Omega)) \sim k^{-\delta/n}.$$

Similar bounds are valid for $H^s(\partial\Omega)$ and on manifolds. (Argument based on Weyl law for eigenvalues.)

If $\partial\Omega$ is real-analytic and $A^R(\partial\Omega)$ is the space of real-analytic functions with uniform Cauchy bounds[1], then

$$e_k(i : A^R(\partial\Omega) \to H^s(\partial\Omega)) \lesssim e^{-ck^{\frac{1}{n}}}.$$

[1] $|\partial^\alpha f(x)| \leq CR^{|\alpha|}\alpha!$ for some $C > 0$

Entropy number bounds

For EIT, need to study spaces of operators.

Theorem. (Spaces of smoothing operators are compressed)

Define the following spaces of operators on $\partial\Omega$:

$$Z^m = \{ T \in B(H^{1/2}, H^{-1/2}) \,:\, T = T^*,\; T(H^{1/2}) \subset H^{-1/2+m} \},$$
$$W^R = \{ T \in B(H^{1/2}, H^{-1/2}) \,:\, T = T^*,\; T(H^{1/2}) \subset A^R \}.$$

Then

$$e_k(i : Z^m \to B(H^{1/2}, H^{-1/2})) \lesssim k^{-\frac{m}{2n}+\delta},$$
$$e_k(i : W^R \to B(H^{1/2}, H^{-1/2})) \lesssim e^{-ck^{\frac{1}{2n-1}}}.$$

Question. Optimality of exponents?

Outline

Instability mechanisms

We will give examples of instability in inverse problems, based on the abstract approach and three instability mechanisms:

1. Strong global smoothing
2. Microlocal smoothing
3. Weak global smoothing

Calderón problem

Let Λ_q be the DN map for $(-\Delta_g + q)u = 0$ in M, where (M, g) is compact with boundary.

Theorem. If M, g, ∂M are real-analytic and

$$\|q_1 - q_2\|_{H^s} \leq \omega(\|\Lambda_{q_1} - \Lambda_{q_1}\|_*), \qquad \|q_j\|_{H^{s+\delta}} \leq 1,$$

with $q_j = 0$ near ∂M, then $\omega(t) \geq c|\log\, t|^{-\sigma}$ for some $\sigma > 0$.

Proof. Take $X_1 = H^{s+\delta}$, $X = H^s$, $Y = B(H^{1/2}, H^{-1/2})$. Need to show that

$$F : q \mapsto \Lambda_q - \Lambda_0$$

is compressing. But if $q = 0$ near ∂M, then $\Lambda_q - \Lambda_0$ is in the strongly compressed space of analytic smoothing operators. (Earlier result: Euclidean ball [Mandache 2001].)

Radon transform with limited data

Let Rf be Radon transform of $f \in H^s_K(\mathbb{R}^2)$, $K \subset \mathbb{R}^2$ compact.

Theorem. If $\mathcal{L} \subsetneq \{\text{lines in } \mathbb{R}^2\}$ is closed, and if

$$\|f\|_{H^s(\mathbb{R}^2)} \leq \omega(\|Rf\|_{H^t(\mathcal{L})}), \qquad \|f\|_{H^{s+\delta}_K} \leq 1,$$

then $\omega(t) \geq c|\log\, t|^{-\sigma}$ for some $\sigma > 0$.

Proof. $\chi_{\mathcal{L}} R$ is microlocally smoothing (it smooths out any singularity near some (x_0, ξ_0)). Testing with wave packets at (x_0, ξ_0) shows that ω cannot be Hölder [Stefanov-Uhlmann 2009]. For our result, use analytic/Gevrey microlocal smoothing of

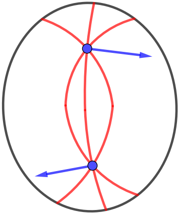
$$\chi_{\mathcal{L}} R : P(H^{s+\delta}) \to H^t$$

where P is a microlocal cutoff. Need Weyl law for microlocally elliptic ΨDOs. Also works for nonlinear IPs.

Geodesic X-ray transform

Let If be the geodesic X-ray transform that integrates f over maximal geodesics in (M, g).

Theorem. Let (M, g) compact, C^∞, strictly convex, nontrapping 2-mfld that has interior conjugate points. If



$$\|f\|_{H^s} \leq \omega(\|If\|_{H^t}), \qquad \|f\|_{H^{s+\delta}} \leq 1,$$

then ω cannot be a Hölder modulus.

Proof. [Monard-Stefanov-Uhlmann 2015]: I has a microlocal kernel. Consider

$$I : P(H^{s+\delta}) \to H^t$$

where P is a projection to the microlocal kernel.

Instability

Smoothing implies instability: F maps into smooth functions / (microlocally) smoothing operators ("compresses distances") $\implies$ IP is strongly unstable. Similar results for unique continuation, backward heat equation, ... if ∂M and the coefficients near (a point of) ∂M are C^∞/real-analytic.

This works for general geometries and variable coefficients. Earlier results for balls, half-spaces, and constant coefficients [Hadamard 1923, John 1960, Mandache 2001, ...].

So far we proved strong instability only if the structures are C^∞/real-analytic. What happens for very rough coefficients? Could the stability improve?

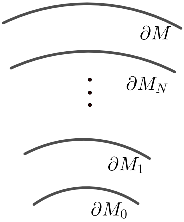
Rough coefficients

Answer: NO. For rough coefficients there is a different compression mechanism ("iterated small regularity gain") $\implies$ instability for Calderón problem etc even with $g \in L^\infty$.

For w solving $\Delta_g w = 0$ near ∂M, factorize

$$\Lambda_q - \Lambda_0 : f \mapsto w|_{\partial M_0} \mapsto w|_{\partial M_1} \mapsto \ldots \mapsto w|_{\partial M_N} \mapsto \partial_\nu w|_{\partial M}$$

where $T_j : w|_{\partial M_j} \mapsto w|_{\partial M_{j+1}}$ has tiny regularity gain (Meyers estimate) but $\|T_j\|$ is large. Estimate entropy numbers of the composition and optimize w.r.t. N.



Summary

- Smoothing (strong/weak/microlocal) implies instability.
- Instability is due to compression properties of forward operator, precisely characterized by entropy numbers.
- Applies to linear and nonlinear inverse problems with general geometries and coefficients.
- Regularization / wave equations / nonlinear PDE?