Bogomolny Equations on $\mathbb{R}^3$ with a Knot Singularity

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The Bogomolny Equations

- Suppose A is an $SU(2)$ connection on $\mathbb{R}^3$ whose curvature is F_A. Suppose is a section of the adjoint $su(2)$ bundle, then the Bogomolny equations are:

$$F_A = \; d_A \; :$$

- Assymototic conditions: $jF_Aj + jd_A \;\; j = O(\frac{1}{r^2})$, $j \;\; j = 1 + O(\frac{1}{r})$.
- Connected components of the configuration space $f(A; \;\;)g$ can be indexed by the "monopole number" k.

  (Classified by the degree of the map $\frac{}{j \;\; j}: S^2 \; ! \; S^2 \quad su(2)$.)

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Donaldson's Description of the Moduli Space

Let M_k be the moduli space of solutions up to $SU(2)$ gauge transformation with monopole number k.

- **Theorem** (Donaldson, 1984) There is a circle bundle $\tilde{M}_k$ over M_k, such that $\tilde{M}_k$ can be identified with rational maps $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ with degree k and $f(\infty) = 0$.
- In particular, $\dim \tilde{M}_k = 4k$, $\dim M_k = 4k - 1$.
- Donaldson's result was based on a study of the moduli space of "Nalm's equations" over $(-1; 1)$ with certain boundary conditions. The relationship between 1-d Nalm's equations and the 3-d Bogomolny equations was established earlier by Nalm and Hitchin.

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Hitchin's Algebraic Geometrical Approach

Hitchin's algebraic geometrical method:

- Consider the moduli space of straight oriented lines in $\mathbb{R}^3$ (identified with $T\mathbb{CP}^1$).
- If an oriented line (with direction v) has $\mathbb{L}^2$ solutions to the ODE equation $r_v^A s + \ s = 0$ on it, then all such lines form an algebraic curve in $T\mathbb{CP}^1$, namely the "spectral curve".
- The spectral curve satisfies certain constraints that can be described algebraically. And verse visa, all such curves correspond to all solutions to the Bogomolny equations.

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Knot Singularity?

- It is trending now to study the gauge theoretic PDEs with a knot singularity. For example
  - Kronheimer and Mrowka's studied the instanton floer theory with a knot singularity;
  - Witten proposed to study Kapustin-Witten equations with certain knot singularity.
- It is natural to ask (and proposed by Taubes), what can we say about the moduli space of the solutions on $\mathbb{R}^3$ with a knot singularity? Can Bogomolny equations with a knot singularity be studied by algebraic geometrical method? This may have the potential to bring knot into algebraic geometry in the future.
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Taubes' Analytical Approach

Taubes have also studied the moduli space of the Bogomolny equations on $\mathbb{R}^3$ using an analytical method in his Ph.D. thesis (Vortices and Monopoles: Structure of Static Gauge Theories).

- Suppose $(A; \ )$ is a configuration. Let L be the linearization of the Bogomolny equations with an extra gauge fixing condition at $(A; \ )$. Let Q be the quadradic term in the Bogomolny equations: $F_A \quad D_A \quad = 0$.
- Let $S = \ {}^0(T \ M) \quad {}^1(T \ M)$ equipped with a "Clifford multiplication ", g be the adjoint $su(2)$ bundle. Then $L : S \quad g \ ! \ S \quad g$ can be written as

$$L = \sum_{j=1}^{3} dx_j \quad r_{A_j} + [\ ;\ ]:$$

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- Ignoring the boundary, integration by part shows that:

$$jL\ \ j^2 = \ \ j\,r_A\ \ j^2 + j[\ ;\ ]j^2 + <\ ;\ [(\ F_A + d_A\ )\wedge\ ]>;$$

$$jL^y\ \ j^2 = \ \ j\,r_A\ \ j^2 + j[\ ;\ ]j^2 + <\ ;\ [(\ F_A\ \ d_A\ )\wedge\ ]>:$$

- Define $k\ \ k_{\mathbb{H}_{(A;\ )}} = (\int_{\mathbb{R}^3} j\,r_A\ \ j^2 + j[\ ;\ ]j^2)^{\frac{1}{2}}$: Immediately,
  - L is Fredholm from $\mathbb{H}_{(A;\ )}$ to $\mathbb{L}^2$.
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- The quadratic part Q is a bounded map from $\mathbb{H}_{(A;\ )}\quad\mathbb{H}_{(A;\ )}$ to $\mathbb{L}^2$.
- A skechy proof:

  Write as $^{=}+\ ^{?}$, where $^{=}=\ ;\ ^{?}\ ?\ .$
  Note that $Q(\ ^{=}_1;\ ^{=}_2)=0$. So $Q(\ _1;\ _2)=Q(\ _1;\ ^{?}_2)$.

  By Sobolev embedding, $_1\ 2\ \mathbb{L}^6$, $^{?}_2\ 2\ \mathbb{L}^6\ \backslash\ \mathbb{L}^2$.

  So $Q(\ _1;\ ^{?}_2)\ _{\mathbb{L}^2}$ is bounded by $k\ _1k_{\mathbb{H}_{(A;\ )}}\ k\ _2k_{\mathbb{H}_{(A;\ )}}$.
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Knot Singularity

- If there is a knot singularity, then the usual integration by part does not go through: the boundary term near the knot doesn't vanish.
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  By examining the scale near the knot, I make the following adaption:

$$k \quad k^2_{\mathbb{H}_{(A; \;);}} = (\quad_{\mathbb{R}^3 nN} \quad j\, r_A \quad j^2 + j[\; ; \;]j^2) + (\quad_{N} \quad (j\, r_A \quad j^2 + j[\; ; \;]j^2)),$$

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- **De nition** If after a gauge transformation, $= (A;\ )$ is close enough to a model solution $= (A\ ;\ )$ near the knot $(\ _N (j r^A (\ )j^2 + j[\ ;\ ]j^2) < +1)$, then it has a knot singularity with monodromy .
  Here $A\ = \ !$ is the flat connection with monodromy ,
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- **Theorem** (Sun20) Suppose $(A; \ )$ satisfies the assymtotic conditions and both F_A and D_A have bounded $\mathbb{L}^2$ norm. Then the only possible singularity is a knot singularity.
- The drawback is: the cokernel of L is not guaranteed to be 0.
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Fredholm Theory

- **Theorem** (Sun20) If $\in (0,\frac{1}{8}) \cup (\frac{3}{8},\frac{1}{2})$, then L is Fredholm. In this situation, the moduli space has a local "analytical structure": Locally it can be identified with the pre-image $f^{-1}(0)$ of a real analytical map between f Euclidean spaces.
- A sketchy proof:

$$\int_N |L \quad|^2 = c\| \quad \|^2_{\mathbb{H}(N \ )} + \int_{\partial N} \text{boundary term } A + (\quad);$$

$$\int_{\mathbb{R}^3 \setminus N} |L \quad|^2 \quad \| \quad \|_{\mathbb{H}_{\mathbb{R}^3 \setminus N}} + \int_{\partial N} \text{boundary term } B + (\quad):$$

Here $\int_{\partial N}(2A+B)$ is compact relative to $\| \quad \|_{\mathbb{H}_{(A; \ );}}$, so

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Similar inequality holds for L^{y}, implying that L is Fredholm.

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Conjecture

- 1. L is also Fredholm when $2\ [\frac{1}{8}; \frac{3}{8}]$.
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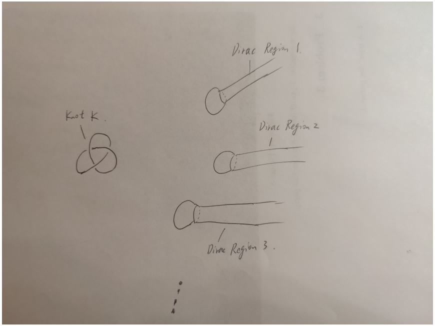

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  A is almost flat and   is almost a constant outside of the Dirac region.
- If the Dirac regions and the knot are far away, then it is possible to "glue" any number of Prasad-Sommerfield monopoles onto the model solution with knot singularity.
- (Sun20) If I shrink the knot to be small enough, then the relative compact term in

$$kL \quad k^2_{\mathbb{L}^2} \quad ck \;\; k^2_{\mathbb{H}_{(A;\ \ );}} \qquad \text{relatively compact term}:$$

  can be bounded $\frac{c}{2}k \;\; k^2_{\mathbb{H}_{(A;\ \ );}}$, which implies that the cokernel of L is 0. (So the moduli space has a manifold structure nearby in this situation.)

Thank you!