Moments of the Riemann zeta function

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University of Lethbridge

Alberta Number Theory Days 13
online/BIRS
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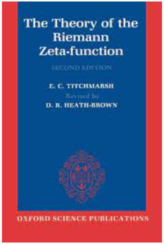
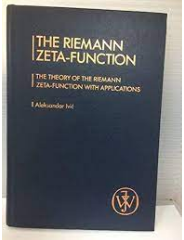
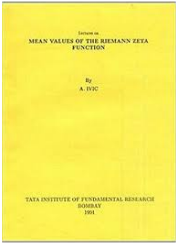
Classic textbooks







E.C. Titchmarsh

D.R. Heath-Brown

A. Ivic

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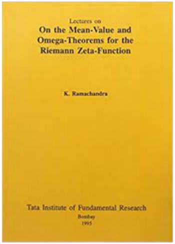

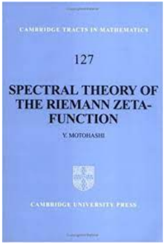

Ramachandra

Motohashi

Riemann zeta function

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- Zeros symmetric about the line $\mathfrak{Re}(s) = \frac{1}{2}$ and the real axis.

Zeros of the Riemann zeta function

Conjecture (Riemann Hypothesis (RH))

All non-trivial zeros of ζ *satisfy* $\mathfrak{Re}(s) = \frac{1}{2}$.

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$\zeta(\sigma + it)$ *has no zeros in the region*

$$\sigma \geq 1 - \frac{1}{R \log t} \text{ and } |t| > 2 \quad \text{with } R = 5.57$$

Former records: de la Vallée Poussin 30.46 (1899), Westphal 17.53 (1938), Stechkin 9.65 (1975), Kadiri 5.69 (2004).

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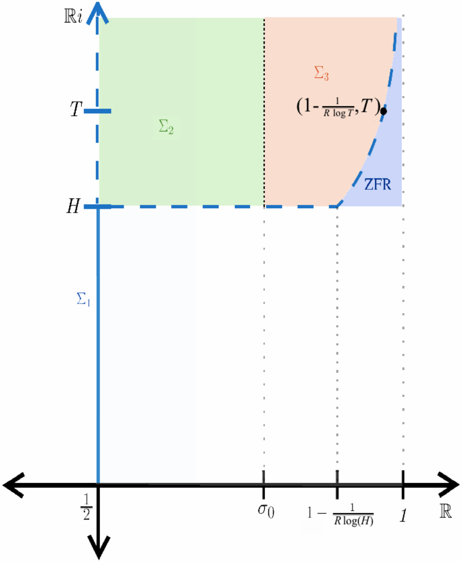
Theorem (Zero density: Kadiri-Lumley-N - 2018)

For $t \ge 1$,

$$\#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ \tfrac{3}{4} \le \beta < 1, |\gamma| \le t\} \le 5.3 t^{\frac{2}{3}} (\log t)^{\frac{7}{2}}.$$

Map of zeros of zeta



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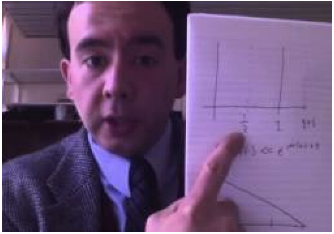
[1] Courtesy of Allysa Lumley

Habiba Kadiri
(Lethbridge)



Tim Trudgian
(UNSW, Lethbridge PDF '10-'12)

Allysa Lumley
(CRM PDF '19-'22,
Lethbridge MSc. '12-'14)

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- The Lindelöf hypothesis is true if and only if

$$I_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon} \text{ for all } k \in \mathbb{N} \text{ and all } \varepsilon > 0.$$

Bounds for $|\zeta(\frac{1}{2}+it)|$

Convexity (easy) bound: $|\zeta(\frac{1}{2}+it)| \ll t^{\frac{1}{4}+\varepsilon}$.

(Phragmén-Lindelöf $|\zeta(-\varepsilon'+it)| \ll |t|^{\frac{1}{2}+\varepsilon'}$ and $|\zeta(1+\varepsilon'+it)| \ll 1 = |t|^0$)

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Subconvexity bound: there exists $\vartheta < \frac{1}{4}$ such that $|\zeta(\frac{1}{2}+it)| \ll t^{\vartheta+\varepsilon}$

Researcher(s)	year	ϑ
Hardy-Littlewood (Weyl)	1921	$\frac{1}{6} = 0.1666\ldots$
Walfisz	1924	$\frac{163}{988} = 0.1649\ldots$
Titchmarsh	1931	$\frac{27}{164} = 0.1646\ldots$
Titchmarsh	1942	$\frac{19}{116} = 0.1637\ldots$
Kolesnik	1982	$\frac{32}{116} = 0.1620\ldots$
Bombieri-Iwaniec	1986	$\frac{9}{56} = 0.1607\ldots$
Huxley	1993	$\frac{89}{154} = 0.15615\ldots$
Huxley	2005	$\frac{32}{205} = 0.15609\ldots$
Bourgain	2014	$\frac{53}{342} = 0.15476\ldots$

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For any real $k > 1$,

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These results suggest

$$I_k(T) \sim C_k T(\log T)^{k^2}.$$

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k-th divisor function:

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$$e_1 + e_2 + \cdots + e_k = j \text{ for } e_i \geq 0 \Longrightarrow \textbf{HW} : d_k(p^j) = \binom{k+j-1}{j}$$

Divisors on average

- Dirichlet's hyperbola trick

$$\sum_{n\le x} d_2(n) = \sum_{n\le x}\sum_{ab=n} 1 = \sum_{ab\le x} 1$$
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- Elementary argument or Perron's formula

$$\sum_{n\leq x} d_k(n)^2 \sim a_k' x(\log x)^{k^2-1} \Longrightarrow \sum_{n\leq x}\frac{d_k(n)^2}{n} \sim \frac{a_k}{(k^2)!}(\log x)^{k^2}.$$

for constants a_k', a_k.

Additive divisor sums

Additive divisor sums **are much harder to evaluate** than $\sum_{n\le x} d_k(n)^2$.

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Why? prime factorization of n not directly related to $n + r$.

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Why? prime factorization of n not directly related to $n+r$.

- $k=\ell=2$. Motohashi 1994, Estermann 1930

$$D_{2,2}(x,r) = x(c_0(r)\log^2(x) + c_1(r)\log(x) + c_2(r)) + O(x^{\frac{2}{3}+\varepsilon})$$

uniformly for $|r| \leq x^{\frac{20}{27}}$ where $c_0(r) = \frac{6}{\pi^2}\sum_{d|r} d^{-1}$.

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- $3 \le k \le 15$, $\ell = 2$. Topacogullari 2018. Drappeau 2017.

$$D_{k,2}(x,r) = x(c_0(r)(\log x)^k + \cdots + c_{k-1}(r) + c_k(r)) + O(x^{\frac{56}{57}+\varepsilon})$$

  uniformly for $|r| \le x^{\frac{15}{19}}$. (There is a result for $k \ge 16$ too.)

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Method: Spectral theory of automorphic forms (Kuznetsov's formula).

Conjecture ($AD_{k,\ell}$: Vinogradov-Ivic-Conrey-Gonek, 1989-1998)

Let $\varepsilon, \varepsilon' > 0$. *We have*

$$\sum_{n \le x} d_k(n) d_\ell(n+r) = \mathcal{M}_{k,\ell}(x,r) + \mathcal{O}(x^{\frac{1}{2}+\varepsilon}),$$

uniformly for $1 \le |r| \le x^{1-\varepsilon}$ *where*

$$\mathcal{M}_{k,\ell}(x,r) = x(c_0(r)(\log x)^{k+\ell-2} + \cdots + c_{k+\ell-3}(r)\log x + c_{k+\ell-2}(r)),$$

The coefficients $c_j(r)$ *have very complicated formulae. We have*

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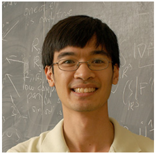
Simple formulae for $c_0(r)$: N.-Thom (2016) and Tao (blogpost, 2016).

Mark Thom (USRA '08,'09)



Terry Tao ("Mozart of Math")

$$D_{k,\ell}(x,r) \sim c_0(r)x(\log x)^{k+\ell-2}, \; c_0(r) \approx \prod_{p|r}\left(1+\frac{(k-1)(\ell-1)}{p}\right)$$

- Versions of $\mathrm{AD}_{k,\ell}$ true if $k \geq 2$, $\ell = 2$. Topacogullari, Motohashi, many others.
- Lower bound: N.-Thom, 2019. Upper bound: Henriot, 2015.

$$c_0(r)x(\log x)^{k+\ell-2} \ll D_{k,\ell}(x,r) \ll c_0(r)x(\log x)^{k+\ell-2} \text{ for } |r| \leq x^A$$

- Almost all results in r: Matomaki, Radziwill, Tao, 2017

Farzad Aryan and Kevin Henriot

Radziwill and Matomaki

Smooth additive divisor sums

When studying $I_k(T)$ we require estimates for smoothed divisor correlations

$$\tilde{D}_{k,\ell}(f,r) = \sum_{m-n=r} \tau_k(m)\tau_\ell(n)f(m,n)$$

where $f : [M,2M] \times [N,2N] \to \mathbb{R}$ is smooth.

- We can formulate a smoothed additive divisor conjecture but its complicated to state.
- Duke-Friedlander-Iwaniec 1994 and Aryan 2017 have proven results towards the smoothed additive divisor conjecture in the case $k = \ell = 2$.

Dirichlet polynomial approximation to $\zeta^k(s)$

Definition

A **Dirichlet polynomial** is a truncation of a Dirichlet series.

eg.

$$D_k(s,N) = \sum_{n=1}^{N} \frac{d_k(n)}{n^s}$$

We expect that $\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \approx D_k(s,N)$ when $s = \frac{1}{2} + it$ and $N = \lfloor t^k \rfloor$.

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$$|\zeta(\tfrac{1}{2}+it)|^{2k} = |\zeta^k(\tfrac{1}{2}+it)|^2 \approx \Big| \sum_{n=1}^{N} \frac{d_k(n)}{n^{\frac{1}{2}+it}} \Big|^2 \text{ where } N = \lfloor t^k \rfloor, \quad \text{by } \textbf{AFE}$$

$$= \Big(\sum_{m=1}^{N} \frac{d_k(m)}{m^{\frac{1}{2}+it}} \Big) \Big(\sum_{n=1}^{N} \frac{d_k(n)}{n^{\frac{1}{2}-it}} \Big) \text{ since } |z|^2 = z\overline{z}.$$

Suggests

$$I_k(T) = \frac{1}{2}\int_{-T}^{T} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \sim \frac{1}{2}\int_{-T}^{T} \sum_{m,n=1}^{N} \frac{d_k(m)d_k(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \Big(\frac{m}{n}\Big)^{-it} dt$$

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$$\int_{-T}^{T} \left(\frac{m}{n}\right)^{-it} dt = \begin{cases} 2T & m = n, \\ 2\frac{\sin(T\log(m/n))}{\log(m/n)} & m \neq n. \end{cases}$$

Diagonal terms $m = n$ terms and off-diagonal terms $m \neq n$:

$$I_k(T) \sim T\sum_{m=1}^{N} \frac{d_k(m)^2}{m} + \sum_{\substack{m,n=1 \\ m\neq n}}^{N} \frac{d_k(m)d_k(n)}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \frac{\sin(T\log(m/n))}{\log(m/n)} \qquad (2)$$

Observation. $\log(m/n)$ **is small if** m **is close to** n**.** Terms with $m = n + r$ and r small contribute. Need asymptotics for

$$\sum_{m\leq x} \frac{d_k(m)^2}{m} \quad \text{and} \quad \sum_{n\leq x} d_k(n)d_k(n+r)$$

All evaluations of $I_k(T)$ with $k = 1, 2, 3, 4$ use a form of (2)

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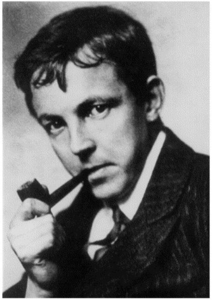
The second moment

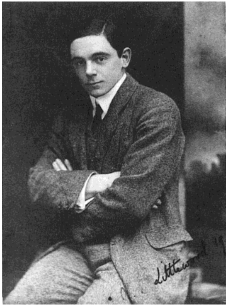
Theorem (Hardy-Littlewood, 1918)

$$I_1(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt \sim T \log T = \frac{1}{1!} \cdot a_1 \cdot T(\log T)$$

where $a_1 = 1$.



G.H. Hardy



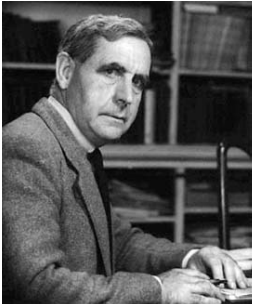
J.E. Littlewood

The fourth moment

Theorem (Ingham, 1926)

$$I_2(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \sim \frac{T}{2\pi^2}(\log T)^4 = \frac{2}{4!}\cdot a_2 \cdot T(\log T)^4$$

where $a_2 = \frac{6}{\pi^2}$.



A.E. Ingham

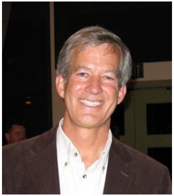
Higher moments ($2k = 6, 8$)

Conjecture (Conrey and Ghosh, 1996)

$$I_3(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^6 \, dt \sim \frac{42}{9!} \cdot a_3 \cdot T(\log T)^9$$

Conjecture (Conrey and Gonek, 1998)

$$I_4(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^8 \, dt \sim \frac{24024}{16!} \cdot a_4 \cdot T(\log T)^{16}$$



J.B. Conrey

A. Ghosh

S.M. Gonek

Higher moments ($2k = 6, 8$)

Conjecture (Keating and Snaith, 1998)

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\, dt \sim \frac{g_k}{(k^2)!} \cdot a_k \cdot T(\log T)^{k^2}$$

where $g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$ *and* $a_k \rightsquigarrow$ *infinite product.*

J. Keating

N. Snaith

$$I_k(T)=\int_0^T |\zeta(\tfrac12+it)|^{2k}\,dt \sim \frac{g_k}{(k^2)!}\cdot a_k\cdot T(\log T)^{k^2}.$$

$$g_k=(k^2)!\prod_{j=0}^{k-1}\frac{j!}{(j+k)!} \text{ and } a_k=\prod_p(1-\tfrac1p)^{k^2}\sum_{j=0}^{\infty}\binom{k+j-1}{j}^2 p^{-j}$$

yr	author(s)	k	g_k	a_k	$\frac{g_k}{(k^2)!}\cdot a_k$	technique
1918	Hardy-Littlewood	1	1	1	1	DP+AFE
1926	Ingham	2	2	$\frac{6}{\pi^2}$	$\frac{1}{2\pi^2}$	DP+AFE
1996	Conrey-Ghosh	3	42			DP+AFE
1998	Conrey-Gonek	3,4	42,24024			DP+AFE
1998	Keating-Snaith	all k				RM
2003	DGH [2]	all k				MDS
2004	CFKRS [3]	all k				AFE

Black =Theorem, **Red= Conjectural method**
AFE = Approximate Functional Equation, DP= Dirichlet Polynomials, RM= Random matrices, MDS = Multiple Dirichlet series
Homework. $g_3=42$, $g_4=24024$, $g_k\in\mathbb{N}$.

[2] DGH=Diaconu, Goldfeld, Hoffstein
[3] CFKRS= Conrey, Farmer, Keating, Rubinstein, Snaith

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Random Matrix Theory history

- 1972: Freeman Dyson and Hugh Montgomery - Statistics of zeros of $\zeta(s)$ are the same as statistics of eigenangles of random matrices

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- Theoretical evidence: Montgomery (1972), Hejhal (1994), Rudnick-Sarnak (1996)
- 1998: Keating and Snaith model $\zeta(s)$ by the characteristic polynomial $Z(U, \theta)$

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- **Average spacing** $= \frac{2\pi}{N}$.
  (N *numbers in an interval of length* 2π)
- Label the zeros of $\zeta(s)$ as $\frac{1}{2} + i\gamma_n$ where

$$\gamma_1 < \gamma_2 < \cdots < \gamma_n \le \gamma_{n+1} \le \cdots .$$

Average spacing $= \frac{2\pi}{\log \gamma_j} \sim \frac{2\pi}{\log T}$ when $T \le \gamma_j \le 2T$.
(*Follows for formula for* $N(T) \sim \frac{T}{2\pi} \log T$.)

Zero and eigenangle statistics

- Normalize the eigenangles $\widehat{\theta}_j = \frac{N}{2\pi}\theta_j$ for $j = 1, \ldots, N$ so that

$$\widehat{\theta}_{j+1} - \widehat{\theta}_j \approx 1 \text{ on average }.$$

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- **Dyson-Montgomery observation**. As $N \to \infty$

$$\# \frac{\{1 \le j \le N \mid \widehat{\theta}_{j+1} - \widehat{\theta}_j \in [a, b]\}}{N}$$

behaves like

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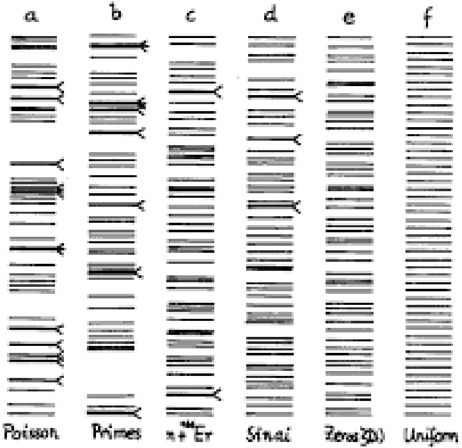
It has been proven there exists a function $p(u)$ such that

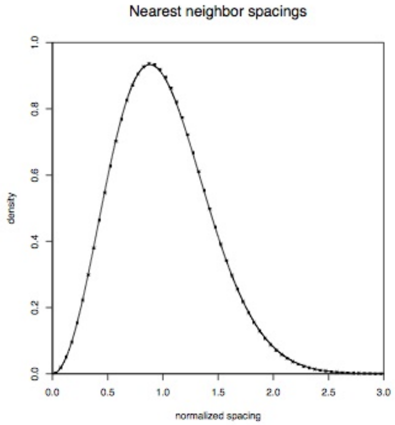
$$\lim_{N\to\infty} \# \frac{1}{N}\{1 \leq j \leq N \mid \widehat{\theta}_{j+1} - \widehat{\theta}_j \in [a, b]\} = \int_a^b p(u)\, du$$

Neighbour Spacings



cit. O. Bohigas and M-J Giannoni, *Chaotic motion and random matrix theories*.

Odlyzko's calculations



Andrew Odlyzko, *On the Distribution of Spacings Between Zeros of the Zeta Function*

- A model for the ζ function

$$Z(U,\theta) = \det(I_n - Ue^{-i\theta}) = \prod_{j=1}^{N}(1 - e^{i(\theta_j - \theta)}).$$

where $U \in U(N)$ and $\theta \in [0, 2\pi]$.

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- The integral

$$\langle |Z(U,\theta)|^{2k} \rangle_{U(N)} = \int_{U(N)} |Z(U,\theta)|^{2k} d\mu_N$$

is a model for

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\, dt$$

Keating and Snaith show that

$$\begin{aligned} \langle |Z(U,\theta)|^{2k} \rangle_{U(N)} &= \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} \\ &= \prod_{j=0}^{k} \frac{j!}{(j+k)!} N^{k^2} + O(N^{k^2-1}) \end{aligned} \tag{3}$$

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$$\langle |Z(U,\theta)|^{2k} \rangle_{U(N)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} = \prod_{j=0}^{k} \frac{j!}{(j+k)!} N^{k^2} + O(N^{k^2-1}) \tag{3}$$

Equating average spacing of zeros/eigenangles: $\frac{2\pi}{\log T} = \frac{2\pi}{N}$

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim a_k \cdot \prod_{j=0}^{k} \frac{j!}{(j+k)!} \cdot (\log T)^{k^2} \tag{4}$$

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Approximate functional equation

- **Classical Approximate Functional Equation**
  Hardy and Littlewood (1921),Riemann (unpublished) - Siegel (?1859?-1932) : $N = (\frac{t}{2\pi})^{\frac{1}{2}}$.

$$\zeta(\tfrac{1}{2}+it) = \sum_{n\le N} \frac{1}{n^{\frac{1}{2}+it}} + f(t)\sum_{n\le N}\frac{1}{n^{\frac{1}{2}-it}} + c_1 t^{-\frac{1}{4}} + c_2 t^{-\frac{5}{4}} + \cdots + O(t^{-(2m+3)/4}).$$

where

$$f(t) \sim e^{\frac{i\pi}{4}}\left(\frac{2\pi e}{t}\right)^{it} \text{ and } |f(t)| = 1.$$

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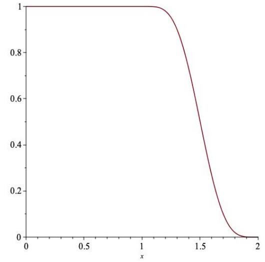
- **Smooth approximate functional for** $\zeta(\frac{1}{2}+it)^k$
  Lavrik (1966): $N = t^{\frac{k}{2}}$

$$\zeta(\tfrac{1}{2}+it)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{\frac{1}{2}+it}} \varphi\left(\frac{n}{N}\right) + f(t)^k \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{\frac{1}{2}-it}} \varphi\left(\frac{n}{N}\right) + O(\exp(-t^2/3))$$

  Smoothing improves error terms!

Smoothing and Approximate Functional Equation for $|\zeta(\frac{1}{2}+it)|^{2k}$



smooth weight φ

- **Smooth approximate functional equations**
  Heath-Brown (1979): $N = t^k$.

$$|\zeta(\tfrac{1}{2}+it)|^{2k} = \sum_{m,n=1}^{\infty} \frac{d_k(m)d_k(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it}\varphi\left(\frac{mn}{N}\right) + O(\exp(-t^2/2))$$

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The sixth moment $I_3(T)$

- AD$_{3,3}$: $\sum_{n\leq x} d_3(n)d_3(n+r) \sim c_{3,3}(r)x\log^4(x)$ with error term $O(x^{0.66})$ for $|r| \leq \sqrt{x}$.

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- The smooth ternary additive divisor conjecture is the analogy of $AD_{3,3}$ for the smoothed sums $\sum_{m-n=r} d_3(m)d_3(n)f(m,n)$ where $f:[M,2M]\times[N,2N]\to\mathbb{R}$ is smooth.
- Conrey and Gonek previously provided a heuristic argument for this.

The eighth moment $I_4(T)$

- $\mathrm{AD}_{4,4}$: $\sum_{n\le x} d_4(n)d_4(n+r) \sim c_{4,4}(r)x\log^{10}(x)$
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Theorem (N-Shen-Wong, 2021+, in preparation)

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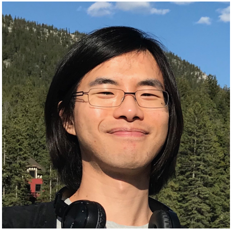
Quanli Shen



Peng-Jie Wong

$I_k(T)$, $k \geq 5$: other divisor sums

- $I_k(T)$ can be modelled in terms of the long Dirichlet polynomials:

$$\int_0^T \Big| \sum_{n \leq T^\eta} \frac{d_k(n)}{n^{\frac{1}{2}+it}} \Big|^2 dt \text{ with } \eta \geq 1. \tag{5}$$

- The case $\eta \in [1,2)$ are conjecturally understood, assuming $\mathrm{AD}_{k,k}$. For $\eta \geq 2$ they are not well understood.
- Conrey and Keating formulated a conjecture for the integrals in (5) and have a program to understand them in terms of:

$$\sum_{\substack{m_1 m_2 \leq x \\ m_1 N - n_1 M = h_1 \\ m_2 M - n_2 N = h_2}} \frac{d_{k_1}(m_1) d_{k_2}(m_2) d_{\ell_1}(n_1) d_{\ell_2}(n_2)}{m_1 m_2} f\Big(\frac{T h_1}{2\pi m_1 N} + \frac{T h_2}{2\pi m_2 N} \Big)$$

where

$$1 \leq M \leq N,\ h_1, h_2 \in \mathbb{Z},\ k_1, k_2, \ell_1, \ell_2 \in \mathbb{N},\ \text{and } f \text{ is smooth.}$$

Mean values of long Dirichlet polynomials

Theorem (N.-Hamieh, 2021)

Let $1 < \eta < \frac{4}{3}$, $N = T^{\eta}$, $L = \log(\frac{t}{2\pi})$, *and* ω *is a smooth weight. Then*

$$\int_{-\infty}^{\infty} \omega(t) \left| \sum_{n=1}^{N} \frac{d_2(n)}{n^{\frac{1}{2}+it}} \right|^2 dt \sim \frac{1}{\zeta(2)} \int_{-\infty}^{\infty} \omega(t) \frac{1}{4!} \Big(-\log^4(N) + 8\log^3(N)L - 24\log^2(N)L^2 + 32\log(N)L^3 - 14L^4 \Big)\, dt$$

with a power savings error term.

Alia Hamieh
UNBC, UL PIMS PDF '15-'17

- Special case of a more general theorem with coefficients $d_k(n)$, assuming smooth form of $\mathrm{AD}_{k,k}$ in the case $\eta \in (1, 2)$.
- This confirms a conjecture of Conrey-Gonek.

Multiple Dirichlet Series

Diaconu, Goldfeld, and Hoffstein introduced the multi-variable complex functions

$$Z(s_1, \ldots, s_{2m}, w) = \int_1^{\infty} \zeta(s_1 + \varepsilon_1 it) \cdots \zeta(s_{2m} + \varepsilon_{2m} it) \left(\frac{2\pi e}{t}\right)^{kit} t^{-w} \qquad (6)$$

where

$$w, s_1, s_2, \ldots, s_{2k} \in \mathbb{C}, \varepsilon_k = \pm 1, 1 \leq k \leq 2m.$$

- They show that $Z(s_1, \ldots, s_{2m}, w)$ satisfies certain quasi functional equations.
- Assuming certain meromorphicity conjectures for $Z(s_1, \ldots, s_{2m}, w)$ they deduce the Keating-Snaith conjecture:

$$I_k(T) \sim \frac{g_k}{(k^2)!} a_k T (\log T)^{k^2}.$$

- They use an unpublished Tauberian theorem due to Stark.

Spectral reciprocity: Motohashi's exact formula

Let ω be a smooth weight function.

$$\begin{aligned}\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it)|^4 \omega(t)\,dt &= \mathrm{Mainterm}(\omega) + \sum_{j=1}^{\infty} \alpha_j L(\tfrac{1}{2}, f_j)^3 \tilde{\omega}(j) \\ &+ \pi \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}+it)|^6}{|\zeta(1+2it)|^2} \widehat{\omega}(t)\,dt + \sum_{k=1}^{\infty} \Big(\sum_{f \in S_{2k}(\Gamma)} L(\tfrac{1}{2}, f)^3 C(k,f,\omega) \Big).\end{aligned} \tag{7}$$

- If $\omega \approx 1_{[T,2T]}$, then $\mathrm{Mainterm}(\omega) \sim \frac{T}{2\pi^2}(\log T)^4$.
- $\{L(s,f_j)\}$ ranges through Maass form L-functions attached to full modular group.
- $\{L(s,f)\}$ ranges through a basis of Hecke eigenforms of $S_{2k}(\Gamma)$.
- $\tilde{\omega}(j)$ and $\widehat{\omega}(t)$ are certain integral transforms of ω.
- This formulae gives the best error term for $I_2(T)$ due to Zavorotnyi (1989): $O(T^{\frac{2}{3}+\varepsilon})$.

Open problems

- Evaluate asymptotically $\sum_{n\le x} d_3(n)d_3(n+1)$.
- Find an asymptotic formula for $\sum_{n\le x} d(n)d(n+1)d(n+2)$.
- Prove that
$$I_3(T) \ge \frac{42}{9!} a_3 T(\log T)^9 (1+o(1)).$$
- Assuming RH, find a good upper bound for $I_3(T)$.
- For some small $\epsilon_0 > 0$ evaluate asymptotically
$$\int_0^T \Big| \sum_{n\le T^{2+\epsilon_0}} \frac{d_k(n)}{n^{\frac{1}{2}+it}} \Big|^2 dt.$$

Even the case $\epsilon_0 = 0$ is open.

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