A (Pandemic Friendly) Handshake with Multi-Vectored Jacobi Forms

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So what's the Goal?

Get to a definition of Multi-Vectored Jacobi Forms with some accompanying motivation for such a definition.

And how are we going to get there?

-Motivated definitions

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-Examples

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-Some early results

Overarching motivation

Reconstruction!

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This is conjectured to be possible in all cases i.e. for all modular tensor categories.

The first definition

Definition

A weakly holomorphic Vector-Valued Modular Form (VVMF) of weight k, multiplier ρ, and rank d is a map $\phi : \mathbb{H} \longrightarrow \mathbb{C}^d$ such that

$$\phi\left(\frac{a\tau + b}{f\tau + g}\right) = \rho(\gamma)(f\tau + g)^k \phi(\tau)$$

$$\forall\, \gamma = \begin{pmatrix} a & b \\ f & g \end{pmatrix} \in SL_2(\mathbb{Z}) + \text{small technical assumptions}$$

Note ρ is a representation of $SL_2(\mathbb{Z})$, and each component ϕ_i is meromorphic in $\mathbb{H}$

What might some examples be?

Example

The Jacobi theta functions $\Theta = (\theta_2, \theta_3, \theta_4)^t$ these obey all the necessary transformation laws including for example

$$\Theta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Theta(\tau).$$

In this case we have weight $\frac{-1}{2}$, rank 3, and multiplier which is consistent with the above

The Second definition

Definition

A weakly holomorphic Jacobi form (JF) of weight k and index m is a map $\varphi : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$\varphi\left(\frac{a\tau + b}{f\tau + g}, \frac{z}{f\tau + g}\right) = (f\tau + g)^k exp\left(2\pi im\frac{fz^2}{f\tau + g}\right)\varphi(\tau, z)$$

$$\varphi(\tau, z + \lambda\tau + \mu) = exp\big(-2\pi im(\lambda^2\tau + 2\lambda z)\big)\varphi(\tau, z)$$

$\forall\, \gamma \in SL_2(\mathbb{Z})$, $(\lambda, \mu) \in \mathbb{Z}^2$ + small technical assumptions

Example again!

Example

The Wierstrass-$\wp$ function of the lattice given by $L = \mathbb{Z} + \tau\mathbb{Z}$, that is,

$$\wp(\tau, z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \mathbb{Z}+\mathbb{Z}\tau \\ \omega \neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

which turns out to be a JF of weight 2 and index 0.

a few preliminaries

Before we define a Multi-Vectored Jacobi Form (MVJF) we need to define a few smaller objects first. Let M be a positive definite lattice of rank r. Additionally let

$$\mathbb{V} := \mathbb{C} \otimes_{\mathbb{Z}} M, \; Jac_M := SL_2(\mathbb{Z}) \ltimes M^2$$

Lastly let $R = (\rho, \rho')$ be a unitary representation of Jac_M on $\mathbb{C}^D$ with T diagonal.

The third and final definition

Definition

A weakly holomorphic MVJF $\mathbb{X}$ of index M, weight k, rank r, multiplier R, and dimension D is a map $\mathbb{X} : \mathbb{H} \times \mathbb{V} \longrightarrow \mathbb{C}^D$ such that

$$\mathbb{X}\left(\frac{a\tau + b}{f\tau + g}, \frac{1}{f\tau + g}\vec{z}\right)(f\tau + g)^{-k} exp\left(-\pi i \frac{f\vec{z} \bullet \vec{z}}{f\tau + g}\right) = \rho(\gamma)\mathbb{X}(\tau, z)$$

$$\mathbb{X}(\tau, \vec{z} + \vec{\lambda}\tau + \vec{\mu}) exp\left(\pi i(\vec{\lambda} \bullet \vec{\lambda}\tau + 2\vec{\lambda} \bullet \vec{z})\right) = \rho'(X)\mathbb{X}(\tau, z)$$

$\forall\, (\gamma, X) \in Jac_M$ + small technical assumptions

Transformation law reminder

Recall

For VVMFs we have

$$\phi\left(\frac{a\tau+b}{f\tau+g}\right)=\rho(\gamma)(f\tau+g)^k\phi(\tau)$$

Recall

For JFs we have

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all VVMFs, are examples of MVJFs by taking $\mathbb{X}(\tau, \vec{0})$ where ρ is given by the multiplier of the VVMF.

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Example

All JFs are MVJFs with $D = \{1\}$, $\rho = \rho' = \mathbb{1}$, and $M = \sqrt{2m}\mathbb{Z}$.

Even more examples (interesting ones this time)

Example

For a positive definite lattice M of rank r then the collection of lattice theta functions $\vec{\Theta}(\tau, \vec{z})$ with components given by

$$\Theta_{[\vec{v}]} = \frac{1}{\left(\eta(\tau)\right)^r} \sum_{\vec{\lambda} \in [\vec{v}]} exp\left(\pi i(\tau \vec{\lambda} \bullet \vec{\lambda} + 2\vec{\lambda} \bullet \vec{z}\right)$$

with a component for each $[\vec{v}] \in {M^\star}/{M}$. Such a MVJF has index M weight 0, rank r, multiplier R (given next), and dimension ${M^\star}/{M}$.

The representation R

The representation $R = (\rho, \rho')$ is given by

$$\rho'(\vec{\lambda}, 0) = \sigma_{\vec{\lambda}},\ \rho'(0, \vec{\mu}) = \left[exp\left(2\pi i \vec{v} \bullet \vec{\mu}\right)\right]_{[\vec{v}],[\vec{v}]}$$

$$\rho(T) = \left[e^{\frac{-2\pi i r}{24}} exp(\pi i \vec{v} \bullet \vec{v})\right]_{[\vec{v}],[\vec{v}]}$$

$$\rho(S) = \left[|{}^{M^\star}\!\!/_{M}|^{\frac{-1}{2}} \exp\left(-2\pi i \vec{\beta} \bullet \vec{\gamma}\right)\right]_{[\beta],[\gamma]}$$

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-Existence of appropriate notions of restriction (sub lattice) and subform (sub representation)
-The "Bridging Theorem"

The inspiration

Theorem (Eichler, Zagier '85)

$\exists$ an isomorphism between the space of JFs of index m, and weight k and the space of VVMFs satisfying certain properties (outlined in their book on page 57-59)

The analogue

Theorem (The "Bridging Theorem" Gannon,G. 2021)

$\exists$ an isomorphism between the space of weakly holomorphic MVJFs of index M, and weight k and the space of VVMFs of weight k satisfying certain technical properties.

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-Working out complete non-trivial examples especially for the characters of VOAs
-Establishing a module structure (or even structures) for these MVJFs
-And more!

For example see $\bigotimes$

For a pair of MVJFs with corresponding weight, rank, index, multiplier, and dimension then we can define

$$\mathbb{X}_1 \otimes \mathbb{X}_2 : \mathbb{H} \times (\mathbb{V}_1 \otimes \mathbb{V}_2) \longrightarrow \mathbb{C}^{D_1 \times D_2}$$

which are defined component wise for $[\vec{v}] = [\vec{\alpha}] \oplus [\vec{\beta}]$, which are given by

$$\mathbb{X}_{[\vec{v}]} = \mathbb{X}_{[\alpha,\beta]} := \mathbb{X}_{[\alpha]}\mathbb{X}_{[\beta]}$$

$\otimes$ continued

$$\mathbb{X}_{[\alpha,\beta]}(\tau,\vec{z}) := \mathbb{X}_{[\alpha]}(\tau,\vec{z}_1)\mathbb{X}_{[\beta]}(\tau,\vec{z}_2), \ \begin{bmatrix}\vec{z}_1\\ \vec{z}_2\end{bmatrix} = \vec{z}$$

where the fact that $M_1 \oplus M_2$ is an orthogonal direct sum allows the whole process to work. Finally resulting in the concrete description...

I promise this is it

$$\mathbb{X}_1 \bigotimes \mathbb{X}_2 := \begin{bmatrix} \mathbb{X}_{[\alpha_1]}\mathbb{X}_2 \\ \vdots \\ \mathbb{X}_{[\alpha_s]}\mathbb{X}_2 \end{bmatrix} = \begin{bmatrix} \mathbb{X}_{[\alpha_1]}\mathbb{X}_{[\beta_1]} \\ \vdots \\ \mathbb{X}_{[\alpha_1]}\mathbb{X}_{[\beta_t]} \\ \mathbb{X}_{[\alpha_2]}\mathbb{X}_{[\beta_1]} \\ \vdots \\ \mathbb{X}_{[\alpha_s]}\mathbb{X}_{[\beta_t]} \end{bmatrix} = \begin{bmatrix} \mathbb{X}_{[\alpha_1,\beta_1]} \\ \vdots \\ \mathbb{X}_{[\alpha_1,\beta_t]} \\ \mathbb{X}_{[\alpha_2,\beta_1]} \\ \vdots \\ \mathbb{X}_{[\alpha_s,\beta_t]} \end{bmatrix}$$

where $\rho_1 \otimes \rho_2$ and $\rho_1' \otimes \rho_2'$ act on $\mathbb{X}_1 \bigotimes \mathbb{X}_2$

Questions?

Thanks for listening!