A slice refinement of Bökstedt periodicity

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Motivation

- BMS constructed filtration on THH and friends, used to build *prismatic cohomology* $\hat{\mathbb{\Delta}}$ + Nygaard filtration $\mathcal{N}^{\geq \bullet}\hat{\mathbb{\Delta}}$.

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- local calculation: $\mathrm{P}_\bullet \mathrm{THH}(R; \mathbb{Z}_p)$ for R perfectoid

Theorem (S.)

Let R be a perfectoid ring. The slice covers/slices of THH *of R are*

$$\mathsf{P}_{2n}\mathrm{THH}(R;\mathbb{Z}_p) = \Sigma^{[n]_\lambda}\mathrm{THH}(R;\mathbb{Z}_p)$$
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If R is p-torsionfree, the RSSS collapses at E_2.

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q-analogues have action of Frobenius:

$$\begin{aligned}\phi(q) &= q^p \\ \phi(f) &= f(q^p) \\ &= f^p + p\delta(f) \\ \phi([n]_q) &= [n]_{q^p}\end{aligned}$$

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q-analogues are ϕ-semimultiplicative:

$$\begin{aligned} [p^3]_q &= \frac{q^{p^3}-1}{q-1} \\ &= \frac{q^p-1}{q-1}\frac{q^{p^2}-1}{q^p-1}\frac{q^{p^3}-1}{q^{p^2}-1} \\ &= [p]_q\phi([p]_q)\phi^2([p]_q) \end{aligned}$$

Roots of unity

Observation: $[p]_q$ is the minimal polynomial of ζ_p.

$$\mathbb{Z}[\zeta_p] = \frac{\mathbb{Z}[q]}{[p]_q} \quad \text{and more generally} \quad \mathbb{Z}[\zeta_{p^n}] = \frac{\mathbb{Z}[q]}{\phi^{n-1}([p]_q)}$$

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Take perfection:

$$\begin{array}{ccccccc}
\mathbb{Z}[q] & \xrightarrow{\phi} & \mathbb{Z}[q] & \xrightarrow{\phi} & \cdots & \longrightarrow & \mathbb{Z}[q^{1/p^\infty}] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}[\zeta_p] & \longrightarrow & \mathbb{Z}[\zeta_{p^2}] & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}[\zeta_{p^\infty}]
\end{array}$$

Witt vectors

Let

$$A = \mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}$$
$$R = \mathbb{Z}_p[\zeta_p^\infty]^\wedge_p = A/[p]_q$$

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Theorem

$$A/[p^n]_q \xrightarrow{\sim} W_n(R) \quad \textit{(A-linear wrt F maps)}$$
$$A/[p^n]_{q^{1/p^n}} \xrightarrow{\sim} W_n(R) \quad \textit{(A-linear wrt R maps)}$$

Notation

Normalization

In p-adic Hodge theory, q is a p-adic analogue of "$e^{2\pi i}$", so we will actually take $R = A/[p]_{q^{1/p}}$.

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From now on:

$$\begin{aligned} R &= \mathbb{Z}_p[\zeta_{p^\infty}]^\wedge_p && \text{or} \quad \text{a perfect } \mathbb{F}_p\text{-algebra } k \\ A &= \mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)} && \text{or} \quad W(k) \\ [p^n]_A &= [p^n]_{q^{1/p}} && \text{or} \quad p^n \end{aligned}$$

$$\mathrm{T}\{\mathrm{HH}, \mathrm{C}^-, \mathrm{P}, \mathrm{C}, \mathrm{F}, \mathrm{R}\} = \mathrm{T}\{\mathrm{HH}, \mathrm{C}^-, \mathrm{P}, \mathrm{F}, \mathrm{R}\}(R; \mathbb{Z}_p)$$

Prism condition

Recall

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Proof.

By $(q-1)$-completeness, it suffices to check $\delta(p) \in \mathbb{Z}_p^\times$. But

$$\delta(p) = 1 - p^{p-1} \in \mathbb{Z}_p^\times.$$

□

Geometric interpretation of the prism condition

Algebraically,

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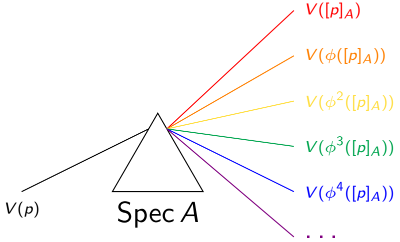
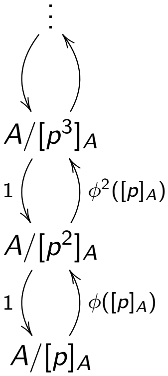


Equivariant interpretation of the prism condition

Define a Mackey functor $\underline{W}$ by



$$\underline{W}(\mathbb{T}/C_{p^n}) = A/[p^{n+1}]_A, \quad \mathrm{tr}_{C_{p^n}}^{C_{p^m}}(x) = \frac{[p^{m+1}]}{[p^{n+1}]}x$$

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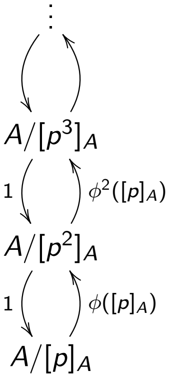

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To get a valid Mackey functor we need

Proposition

For $i \leq j$ there is a congruence

$$\phi^i([p^{j-i}]_A) \equiv up^{j-i} \bmod [p^i]_A$$

for some unit $u \in A^\times$.

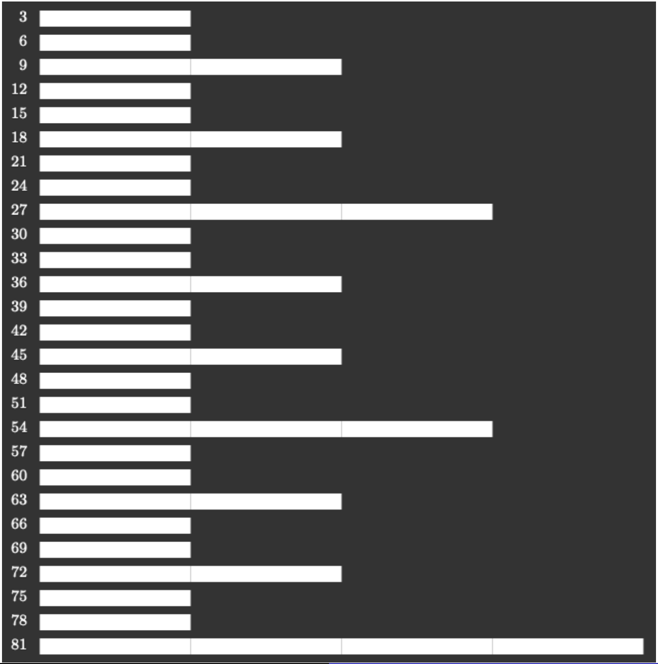
and this is an elaboration of the prism condition.

Legendre's formula

Proposition (Legendre).

The p-adic valuation of a factorial is given by

$$v_p(n!) = \sum_{r=1}^{\infty} \left\lfloor \frac{n}{p^r} \right\rfloor$$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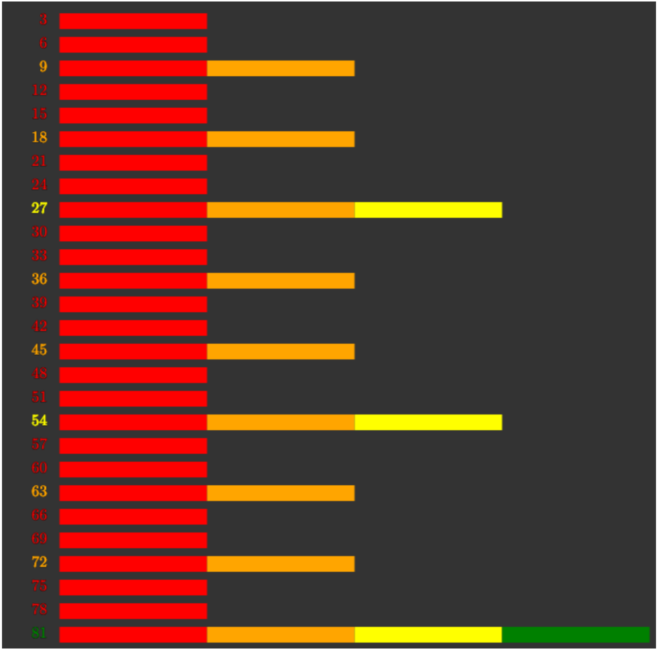
Lemma (Anschütz–le-Bras)

$$\begin{aligned} [n]_q! &= u \prod_{r=1}^{\infty} \phi^{r-1}([p]_q)^{\lfloor n/p^r \rfloor} \\ &= u \prod_{r=1}^{\infty} [p^r]_q^{\lfloor n/p^r \rfloor - \lfloor n/p^{r+1} \rfloor} \end{aligned}$$

for some unit $u \in \mathbb{Z}_p[\![q-1]\!]^\times$.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Circle representations

Complex representations of $\mathbb{T}$ are

$$R(\mathbb{T}) = \mathbb{Z}[\lambda^{\pm}]$$

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We also set

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$$\operatorname{Res}^{\mathbb{T}}_{C_{n+1}}\{n\}_\lambda = \mathbb{C} \otimes_{\mathbb{R}} \bar{\rho}_{C_{n+1}}$$

When working p-locally we have $S^{\lambda^i} \simeq S^{\lambda^j}$ iff $v_p(i) = v_p(j)$, so it suffices to consider

$$\lambda_i := \lambda^{p^i} \qquad i = 0, 1, \ldots$$
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If

$$\alpha = k_0\lambda_0 + \ldots + k_n\lambda_n + k_\infty\lambda_\infty$$

then

$$d_r(\alpha) = \sum_{i \geq r} k_i \quad \text{(includes } i = \infty\text{)}$$
$$k_r(\alpha) = d_r(\alpha) - d_{r+1}(\alpha)$$

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A useful formula is

$$\left\lceil \frac{n}{p^r} \right\rceil - 1 = \left\lfloor \frac{n-1}{p^r} \right\rfloor$$

Tate cohomology

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Slogan

A q-deformation is a deformation from a trivial action to a nontrivial action. Multiplication by n gets deformed to a transfer for a subgroup of index n. (Note $p = 0 \iff \zeta_p = 1$.)

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- Cofiber sequence:

$$\mathrm{P}_n X \to X \to \mathrm{P}^{n-1} X$$

and the *n-slice* of X is

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- When $G = \mathbb{T}$ (or C_{p^∞}), we interpret the slice filtration so that it restricts to the slice filtration for all finite subgroups.

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- recognition criterion for slice tower

Computational facts

$$\mathrm{TF}_* = A[\sigma] \qquad \mathrm{TF}_* = A[t^{-1}]$$

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By Bökstedt periodicity, the Whitehead tower of THH is

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Guess

$$\mathrm{P}_{2n}\mathrm{THH} = \Sigma^{V_n}\mathrm{THH}, \quad V_n \in R(\mathbb{T}), \quad \dim_{\mathbb{C}} V_n = n$$

connected by $R(\mathbb{T})$-graded classes which reduce to σ.

Example with $p = 3$.

$2n$	V_n	$\lceil \frac{2n}{1} \rceil$	$\lceil \frac{2n}{3} \rceil$	$\lceil \frac{2n}{9} \rceil$	...
2		2	1	1	...
4		4	2	1	...
6		6	2	1	...
8		8	3	1	...
10		10	4	1	...

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The slice covers of THH *are given by*

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Euler classes

Let V be a G-representation. Inclusion of zero subspace gives

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- a_V is killed by restriction to G_V
- when $G = \mathbb{T}$, $S^{\infty\lambda_0} = \widetilde{E\mathbb{T}}$, so

  isotropy separation square = arithmetic square for a_{λ_0}.

Cell structures

Cell structures: for $G = C_{p^n}$,

$$
\begin{array}{ccccc}
S^0 & \longrightarrow & S^{\succ} & \longrightarrow & S^{\lambda_r} \\
 & & \downarrow & & \downarrow \\
 & & S^1 \otimes G/C_{p^r+} & \longrightarrow & (\dots) \\
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Hill: for $G = \mathbb{T}$,

$$
\begin{array}{ccc}
\mathbb{T}/C_{p^r+} & \longrightarrow & S^0 \\
 & & \Big\downarrow{\scriptstyle a_{\lambda_r}} \\
 & & S^{\lambda_r}
\end{array}
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Thom classes

Borel homotopy depends only on coarse equivariance:

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Similarly for $\mathrm{THH}_{h\mathbb{T}}$ and TP; expresses that $RO(\mathbb{T})$-grading is not a thing for coarse G-spectra.

Isotropy separation revisited

$$\begin{array}{ccccc} \Sigma X_h & \longrightarrow & X & \longrightarrow & X^\Phi \\ \| & & \downarrow & & \downarrow \\ \Sigma X_h & \longrightarrow & X^h & \longrightarrow & X^t \end{array}$$

a_{λ_0}-periodic, u_{λ_i}-periodic

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Proposition (HHR.; "gold relation")

For $i \leq j$,

$$a_{\lambda_j} u_{\lambda_i} = p^{j-i} a_{\lambda_i} u_{\lambda_j} \text{ in } \underline{\pi}_{\bigstar} \underline{\mathbb{Z}}$$

Tsalidis' theorem $\implies \mathrm{TF}_{\lambda_i} \xrightarrow{\sim} \mathrm{TC}^-_{\lambda_i} = A\langle \sigma u_{\lambda_i}^{-1} \rangle.$

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and for $i \leq j$,

$$\begin{aligned} a_{\lambda_j} u_{\lambda_i} &= \frac{[p^{j+1}]_A}{[p^{i+1}]_A} a_{\lambda_i} u_{\lambda_j} \\ &= \mathrm{tr}_{C_{p^i}}^{C_{p^j}}(1) a_{\lambda_i} u_{\lambda_j} \\ &= \phi^{i+1}([p^{j-i}]_A) a_{\lambda_i} u_{\lambda_j} \end{aligned}$$

Proposition

For $j < n$,

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Proof.

Blue classes predicted by Tsalidis' theorem. For $* = -2$, C_{p^n} cell structure gives

$$0 \longrightarrow \mathrm{TR}^{j+1}_0 \xrightarrow{V^{n-j}} \mathrm{TR}^{n+1}_{\lambda_j - 2} \xrightarrow{a_{\lambda_j}} \mathrm{TR}^{n+1}_{-2} \xrightarrow{F^{n-j}} \mathrm{TR}^{j+1}_{-2}$$

where $\mathrm{TR}^{n+1}_{-2} = 0$ and $\mathrm{TR}^{j+1}_{-2} = 0$.

□

Let $\mathrm{tr}_n \underline{W}$ be the subMackey functor of $\underline{W}$ generated by $\downarrow^{\mathbb{T}}_{C_{p^n}} \underline{W}$, and let $\Phi^n \underline{W}$ be the cofiber:

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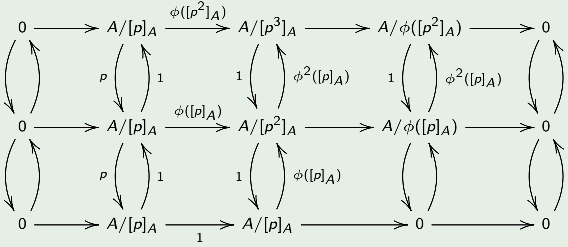

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Proposition. (S.)

The region of $\mathrm{TF}_\bigstar$ where $\bigstar = k - V$ is an integer minus an actual representation is given by

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This refines results of Hesselholt-Madsen / Angeltveit-Gerhardt.

The slice filtration

Slice filtration is

$$\mathsf{F}^{2j}\,\underline{\pi}_{2i}\mathrm{THH} = \mathrm{im}\left(\underline{\pi}_{2i}\mathsf{P}_{2(i+j)}\mathrm{THH} \xrightarrow{\sigma^n u_{\{n\}\lambda}^{-1}} \underline{\pi}_{2i}\mathrm{THH}\right)$$

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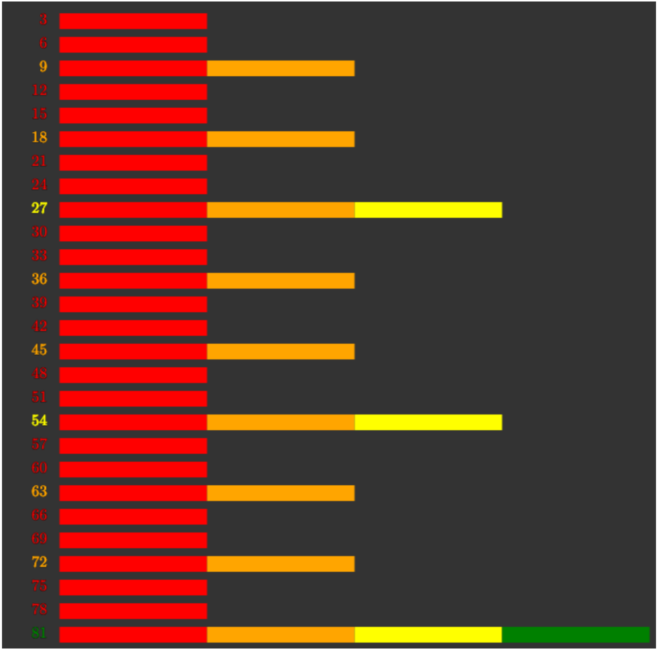
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Otherwise,

$$\mathsf{F}^{2j}\,\underline{\pi}_{2i}\mathrm{THH} = \frac{[p(i+j-1)]_A!}{[p^r]_A^{i-1}\phi^r\left(\left[\left\lfloor\frac{i+j-1}{p^r}\right\rfloor\right]_A!\right)}\underline{\pi}_{2i}\mathrm{THH}.$$

where $r = \left\lceil \log_p\left(\frac{i+j}{i}\right)\right\rceil$.

E_2 page of the slice spectral sequence:

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$$\underline{\pi}_{2i} \mathrm{P}^{2n}_{2n} \mathrm{THH} = \begin{cases} \underline{W} & 0 = i = n \\ \underline{R} & 0 < i = n \\ \Phi^{C_{p^m}} \underline{W}/[p^{h+1}]_A & 0 < i < n \end{cases}$$

where

$$m = \left\lceil \log_p(n/i) \right\rceil - 1$$

$$h = \begin{cases} \min\{v_p(n), \left\lfloor \log_p(n/i) \right\rfloor\} & n/i \text{ not a power of } p \\ \left\lfloor \log_p(n/i) \right\rfloor & n/i \text{ a power of p} \end{cases}$$

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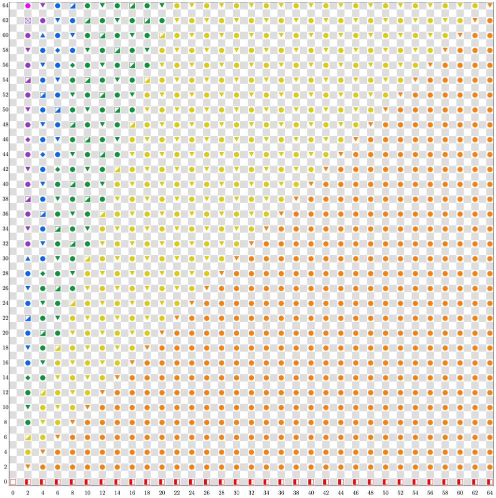
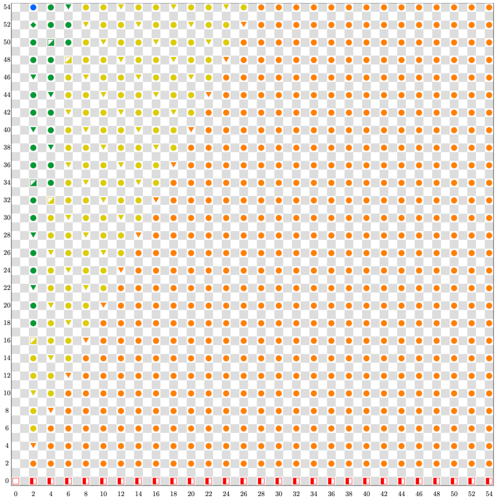
$$h = \begin{cases} \min\{v_p(n), \lfloor \log_p(n/i) \rfloor\} & n/i \text{ not a power of } p \\ \lfloor \log_p(n/i) \rfloor & n/i \text{ a power of p} \end{cases}$$

If R is a perfect $\mathbb{F}_p$-algebra, then

$$\underline{\pi}_{2i+1} \mathrm{P}^{2n}_{2n} \mathrm{THH}(R; \mathbb{Z}_p) = \begin{cases} \mathrm{tr}_{C_{p^{m+h+1}}} \Phi^{C_{p^m}} \underline{W} & n/i \text{ not a power of } p \\ \mathrm{tr}_{C_{p^{m+h+1}}} \Phi^{C_{p^{m+1}}} \underline{W} & n/i \text{ a power of p} \end{cases}$$



E_2 page of the RSSS for $\mathrm{THH}(\mathbb{Z}_2^{\mathrm{cycl}}; \mathbb{Z}_2)$



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