The Geography problem on 4-manifolds: $\frac{10}{8} + 4$

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  2. The Kirby–Siebenmann invariant $ks(N) \in H^4(N;\mathbb{Z}/2) = \mathbb{Z}/2$.

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   $\implies$ *any* $(Q, \ \mathbb{Z}/2)$ *can be realized*

3. *Bilinear form Q: even*
   $\implies$ *only* $\left(Q, \ \frac{\mathrm{sign}(Q)}{8} \mod 2\right)$ *can be realized*

Smooth category

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- $+$ Freedman's theorem:

Theorem

Two closed simply connected smooth 4-manifolds are homeomorphic if and only if they have isomorphic intersection forms.

Two questions

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Question (Botany Problem)

How many non-diffeomorphic 4-manifolds can realize Q?

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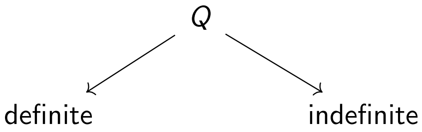

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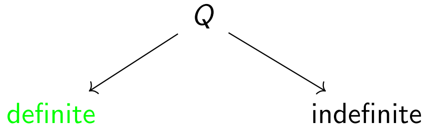

Donaldson's Diagonalizability Theorem

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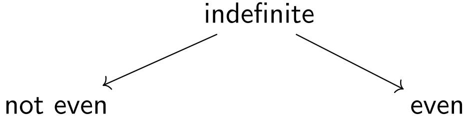
Completely answers the Geography Problem when Q is definite

Indefinite forms



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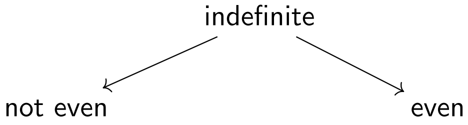

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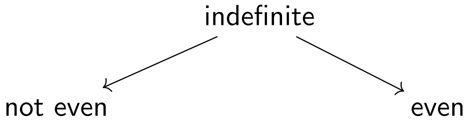
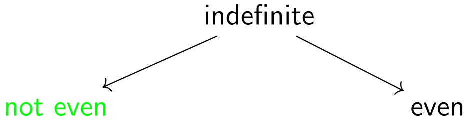


Theorem (Hasse–Minkowski)

1. *Q: not even*
   $Q \cong$ diagonal form with entries ± 1.
2. *Q: even*
   $Q \cong kE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for some $k \in \mathbb{Z}$ and $q \in \mathbb{N}$.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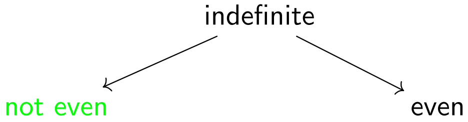
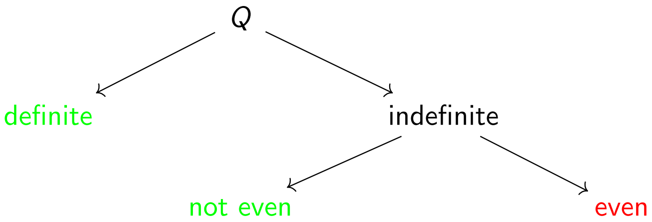


Fact

Q: not even

Q can be realized by a connected sum of copies of $\mathbb{C}P^2$ *and* $\overline{\mathbb{C}P^2}$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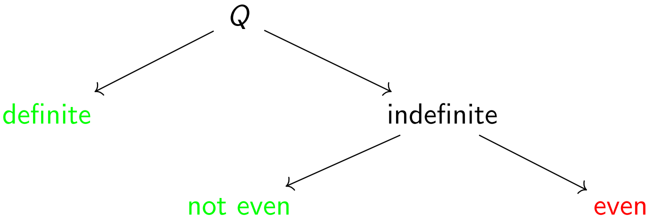

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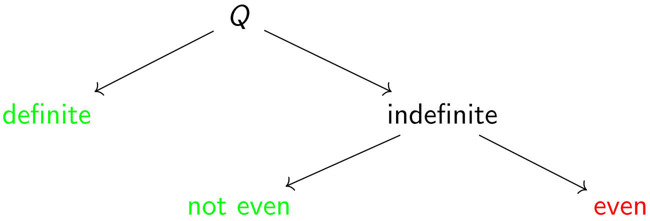

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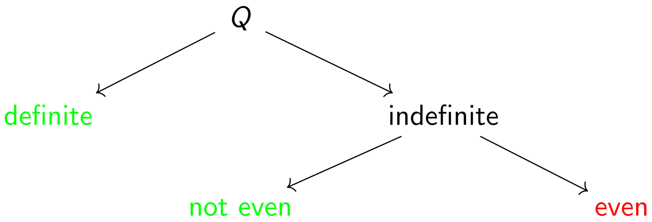

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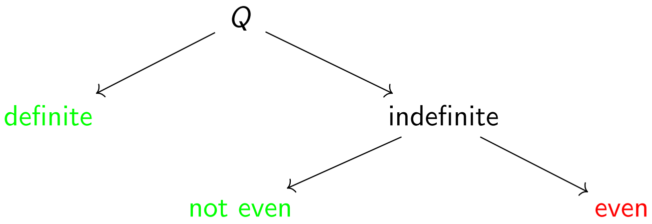

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- By reversing the orientation of M, may assume $k \geq 0$

The $\frac{11}{8}$-Conjecture

Conjecture (version 1)

The form

$$2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

can be realized as the intersection form of a closed smooth spin 4-manifold if and only if $q \geq 3p$.

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  - K_3: $2E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  - $S^2 \times S^2$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The $\frac{11}{8}$-Conjecture

The "only if" part can be reformulated as follows:

Conjecture (version 2)

Any closed smooth spin 4-manifold M must satisfy the inequality

$$b_2(M) \geq \frac{11}{8} |\operatorname{sign}(M)|,$$

where $b_2(M)$ and $\operatorname{sign}(M)$ are the second Betti number and the signature of M, respectively.

Progress on the $\frac{11}{8}$-Conjecture

- $p = 1$, assuming $H_1(M;\mathbb{Z})$ has no 2-torsions: Donaldson (anti-self-dual Yang–Mills equations)

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- Furuta's idea: combined Kronheimer's approach with "finite dimensional approximation"
  - Attacked the conjecture by using Pin(2)-equivariant stable homotopy theory

Furuta's $\frac{10}{8}$-Theorem

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Theorem (Furuta)

For $p \geq 1$, the bilinear form

$$2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is spin realizable only if $q \geq 2p + 1$.

Furuta's $\frac{10}{8}$-Theorem

Corollary (Furuta)

Any closed simply connected smooth spin 4-manifold M that is not homeomorphic to S^4 must satisfy the inequality

$$b_2(M) \geq \frac{10}{8}|\operatorname{sign}(M)| + 2.$$

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The inequality of manifolds with boundaries are proved by Manolescu, and Furuta–Li.

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Theorem (Hopkins–Lin–Shi–X.)

For $p \geq 2$, if the bilinear form $2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is spin realizable, then

$$q \geq \begin{cases} 2p+2 & p \equiv 1,2,5,6 \pmod 8 \\ 2p+3 & p \equiv 3,4,7 \pmod 8 \\ 2p+4 & p \equiv 0 \pmod 8. \end{cases}$$

The limit is $\frac{10}{8} + 4$

Corollary (Hopkins–Lin–Shi–X.)

Any closed simply connected smooth spin 4-manifold M that is not homeomorphic to S^4, $S^2 \times S^2$, or $K3$ must satisfy the inequality

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Furthermore, we show this is the **limit** of the current known approaches to the $\frac{11}{8}$-Conjecture

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- Sobolev completion $\Longrightarrow \widetilde{SW} : H_1 \longrightarrow H_2$ (Seiberg–Witten map)

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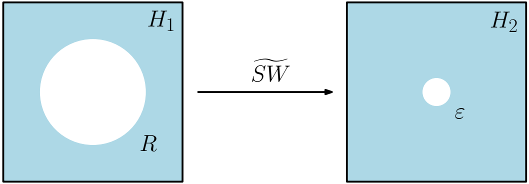
  3. $\widetilde{SW}$ maps $H_1 \setminus \mathring{B}(H_1, R)$ to $H_2 \setminus \mathring{B}(H_2, \varepsilon)$



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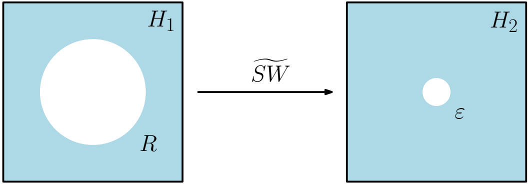

- $S^{H_1} = H_1/(H_1 \setminus \mathring{B}(H_1, R))$
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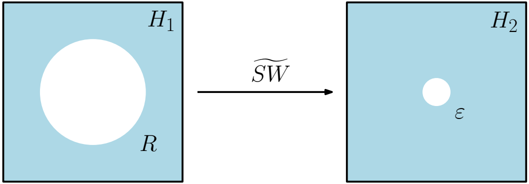

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- $\widetilde{SW}$ induces a Pin(2)-equivariant map between spheres

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- In order to use homotopy theory, we want maps between finite dimensional spheres

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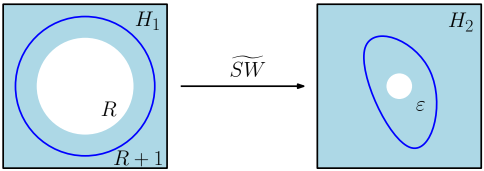

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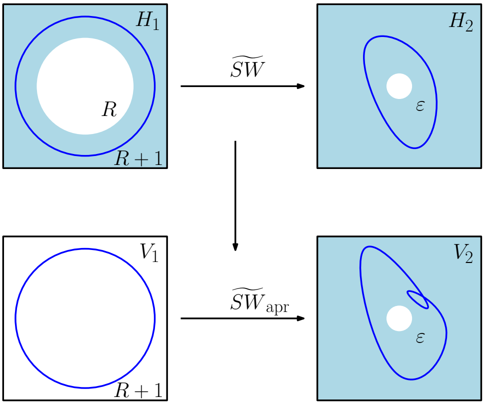
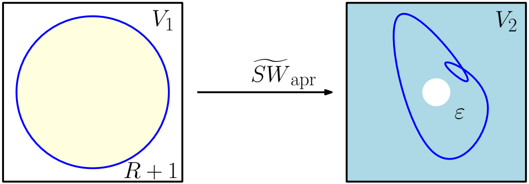
  3. When V_2 is large enough,
     $\widetilde{SW}_{\text{apr}}$ maps $S(V_1, R+1)$ to $V_2 \setminus \mathring{B}(V_2, \varepsilon)$

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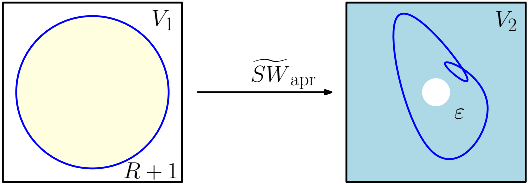

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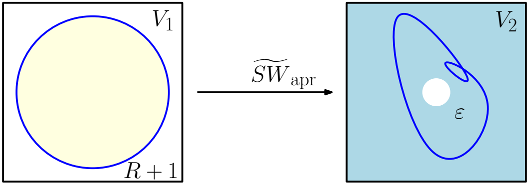

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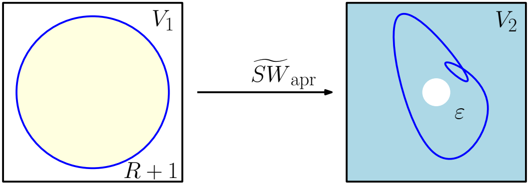

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$$\widetilde{SW}^{+}_{\mathrm{apr}} : S^{V_1} \longrightarrow S^{V_2}$$

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$$

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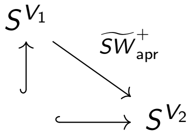

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Proposition (Furuta)

If the intersection form of the manifold M is $2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *then*

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Furuta–Mahowald class

Definition

For $p \geq 1$, a Furuta–Mahowald class of level-(p, q) is a stable map

$$\gamma : S^{p\mathbb{H}} \longrightarrow S^{q\widetilde{\mathbb{R}}}$$

that fits into the diagram

$$\begin{array}{ccc} S^{p\mathbb{H}} & & \\ {\scriptstyle a_{\mathbb{H}}^{p}}\uparrow & \searrow^{\gamma} & \\ S^{0} & \xrightarrow[a_{\widetilde{\mathbb{R}}}^{q}]{} & S^{q\widetilde{\mathbb{R}}} \end{array}$$

- $a_{\mathbb{H}} : S^0 \longrightarrow S^{\mathbb{H}}$
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Theorem (Furuta)

If the bilinear form $2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *is spin realizable, then there exists a level-*(p, q) *Furuta–Mahowald class.*

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Theorem (Furuta)

A level-(p, q) *Furuta–Mahowald class exists* **only if** $q \geq 2p + 1$.

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What is the necessary and sufficient condition for the existence of a level-(p, q) Furuta–Mahowald class?

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- Subsequently, he made a conjecture

Jones' conjecture

Conjecture (Jones)

For $p \geq 2$, a level-(p, q) Furuta–Mahowald class exists **if and only if**

$$q \geq \begin{cases} 2p+2 & p \equiv 1 \pmod 4 \\ 2p+2 & p \equiv 2 \pmod 4 \\ 2p+3 & p \equiv 3 \pmod 4 \\ 2p+4 & p \equiv 0 \pmod 4. \end{cases}$$

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Theorem (Furuta–Kamitani)

For $p \geq 2$, a level-(p,q) Furuta–Mahowald class exists **only if**

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- We completely resolve this question

Main Theorem

Theorem (Hopkins–Lin–Shi–X.)

For $p \geq 2$, a level-(p, q) Furuta–Mahowald class exists **if and only if**

$$q \geq \begin{cases} 2p+2 & p \equiv 1, 2, 5, 6 \pmod 8 \\ 2p+3 & p \equiv 3, 4, 7 \pmod 8 \\ 2p+4 & p \equiv 0 \pmod 8. \end{cases}$$

Comparison of known results

Minimal q such that a level-(p, q) Furuta–Mahowald class exists:

Jones' conjecture	Our theorem	Furuta–Kamitani	
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Any closed simply connected smooth spin 4-manifold M that is not homeomorphic to S^4, $S^2 \times S^2$, or $K3$ must satisfy the inequality

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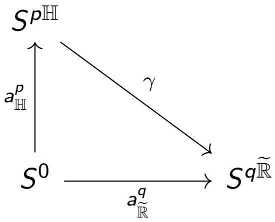
In the sense of classifying all Furuta–Mahowald classes of level-(p, q), this is the **limit**

Furuta–Mahowald classes



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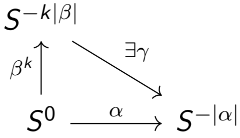

Equivalent Mahowald invariant

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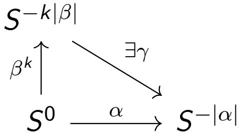
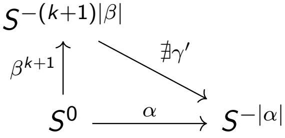
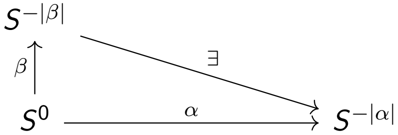




$$S^0 \xrightarrow{\quad\quad\quad\quad \alpha \quad\quad\quad\quad} S^{-|\alpha|}$$


$S^{-|\beta|}$
β
$\exists$
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α
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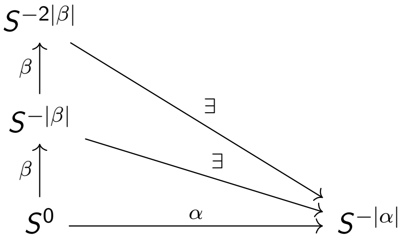
$S^{-2|\beta|}$
β
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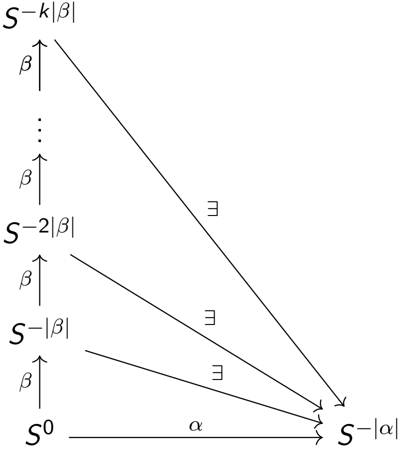

$S^{-k|\beta|}$
β
$\vdots$
β
$S^{-2|\beta|}$
β
$S^{-|\beta|}$
β
S^0
α
$S^{-|\alpha|}$
$\exists$
$\exists$
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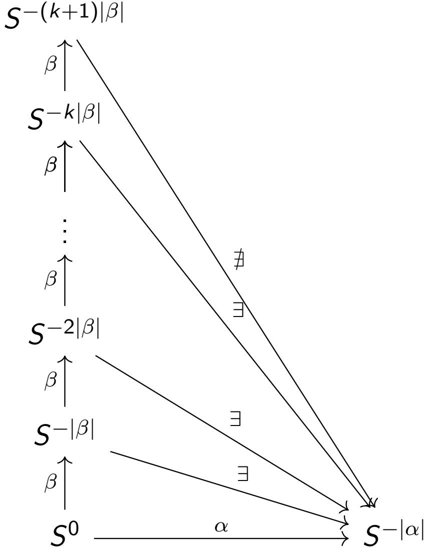

$S^{-(k+1)|\beta|}$
β
$S^{-k|\beta|}$
β
$\vdots$
β
$S^{-2|\beta|}$
β
$S^{-|\beta|}$
β
S^0
α
$S^{-|\alpha|}$
$\nexists$
$\exists$
$\exists$
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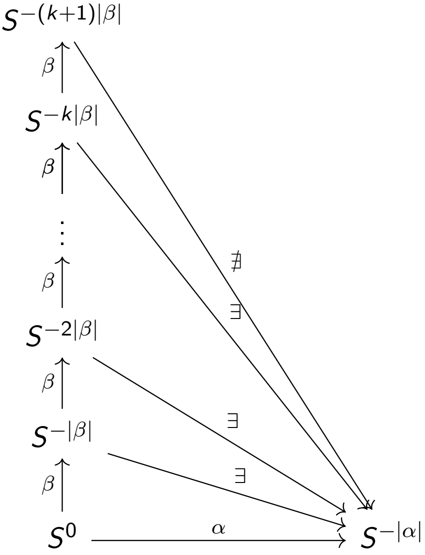

$$\Longrightarrow |M_\beta^G(\alpha)| - |\alpha| = -k|\beta|$$

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For all $q \geq 0$, the $RO(C_2)$-degree of $M^{C_2}_{a_\sigma}(a^q_\sigma)$ is zero.

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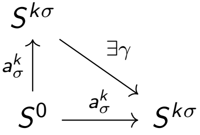

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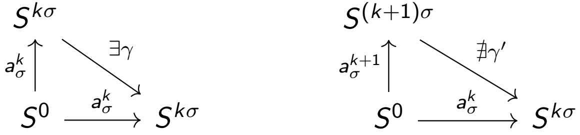

Classical Mahowald invariant – Bruner-Greenlees

$$
\begin{array}{ccc}
S^n & \longrightarrow & S^0 \\
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$$\begin{array}{ccc} S^n & \longrightarrow & S^0 \\ & \Big\Downarrow \Phi^{C_2} & \\ S^n & \xrightarrow{\alpha} & S^0 \end{array}$$

- $\alpha \in \pi_n S^0$

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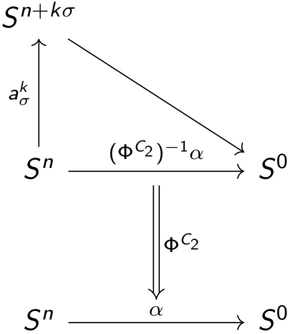

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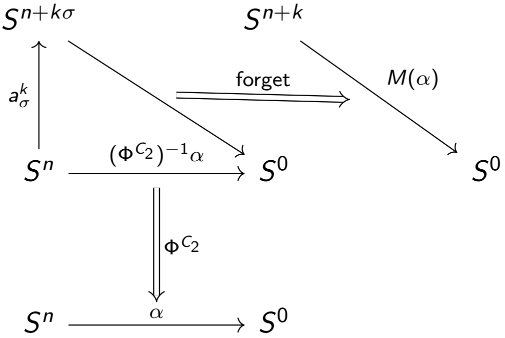

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- Forget to the non-equivariant world $\Longrightarrow$ classical Mahowald invariant $M(\alpha)$

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For $q \geq 1$, the set $M(2^q)$ contains the first nonzero element of Adams filtration q in positive degree.
Moreover, the following 4-periodic result holds:

$$|M^{C_2}_{a_\sigma}\big((\Phi^{C_2})^{-1}2^q\big)| = \begin{cases} (8k+1)\sigma & \text{if } q = 4k+1 \\ (8k+2)\sigma & \text{if } q = 4k+2 \\ (8k+3)\sigma & \text{if } q = 4k+3 \\ (8k+7)\sigma & \text{if } q = 4k+4. \end{cases}$$

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  - λ: 2-dimensional, rotation by 90°
- Crabb, Schmidt, and Stolz studied the C_4-equivariant Mahowald invariant of powers of a_σ with respect to $a_{2\lambda}$

Theorem (Crabb, Schmidt, Stolz)

For $q \geq 1$, the following 8-periodic result holds:

$$|M^{C_4}_{a_{2\lambda}}(a^q_\sigma)| + q\sigma = \begin{cases} 8k\lambda & \text{if } q = 8k+1 \\ 8k\lambda & \text{if } q = 8k+2 \\ (8k+2)\lambda & \text{if } q = 8k+3 \\ (8k+2)\lambda & \text{if } q = 8k+4 \\ (8k+2)\lambda & \text{if } q = 8k+5 \\ (8k+4)\lambda & \text{if } q = 8k+6 \\ (8k+4)\lambda & \text{if } q = 8k+7 \\ (8k+4)\lambda & \text{if } q = 8k+8. \end{cases}$$

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- C_4 is a subgroup of Pin(2)
- Minami and Schmidt used this theorem to deduce the nonexistence of certain Furuta–Mahowald classes

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- To prove our main theorem, we analyze the Pin(2)-equivariant Mahowald invariants of powers of $a_{\widetilde{\mathbb{R}}}$ with respect to $a_{\mathbb{H}}$

Main Theorem

Theorem (Hopkins–Lin–Shi–X.)

For $q \geq 4$, the following 16-periodic result holds:

$$
|M_{a_{\mathbb{H}}}^{\mathrm{Pin}(2)}(a_{\widetilde{\mathbb{R}}}^{q})| + q\widetilde{\mathbb{R}}
= \begin{cases}
(8k-1)\mathbb{H} & \text{if } q = 16k+1 \\
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\end{cases}
\quad
\begin{array}{|ll}
(8k+3)\mathbb{H} & \text{if } q = 16k+9 \\
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(8k+4)\mathbb{H} & \text{if } q = 16k+11 \\
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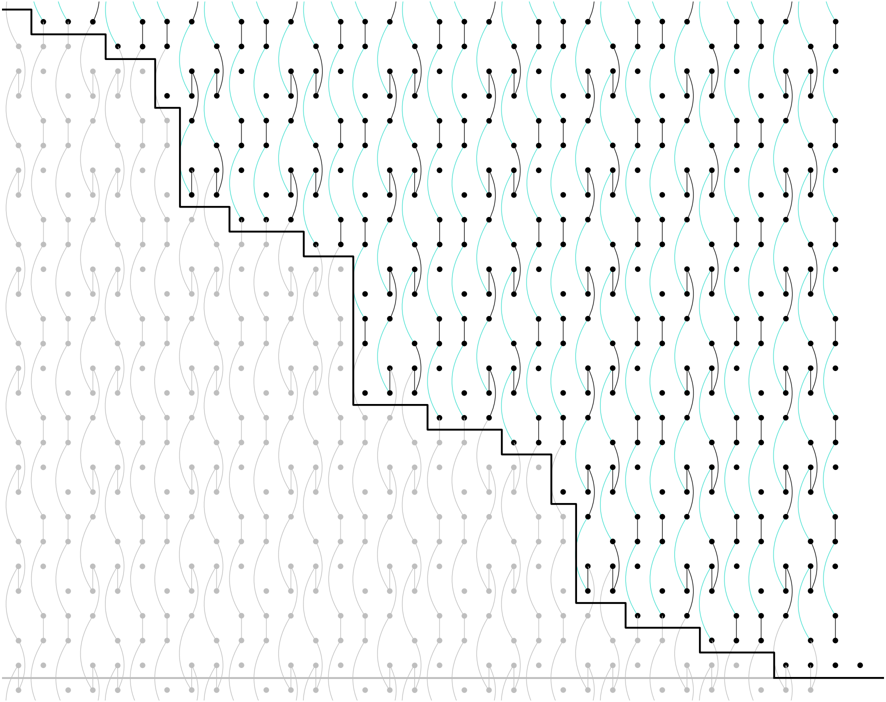
- Had it been $(8k+3)\mathbb{H}$ instead, our result would be 8-periodic
- Jone's conjecture would be true



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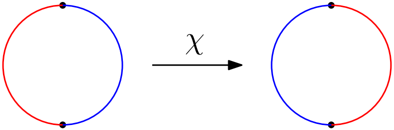

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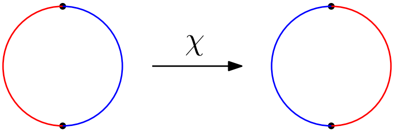

  - C_2-equivariant map

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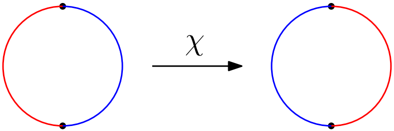

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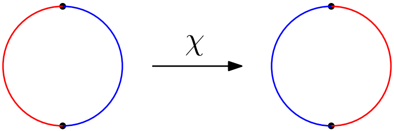

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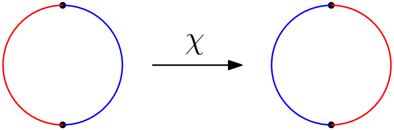

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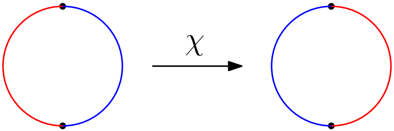

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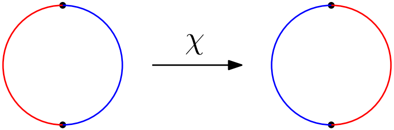

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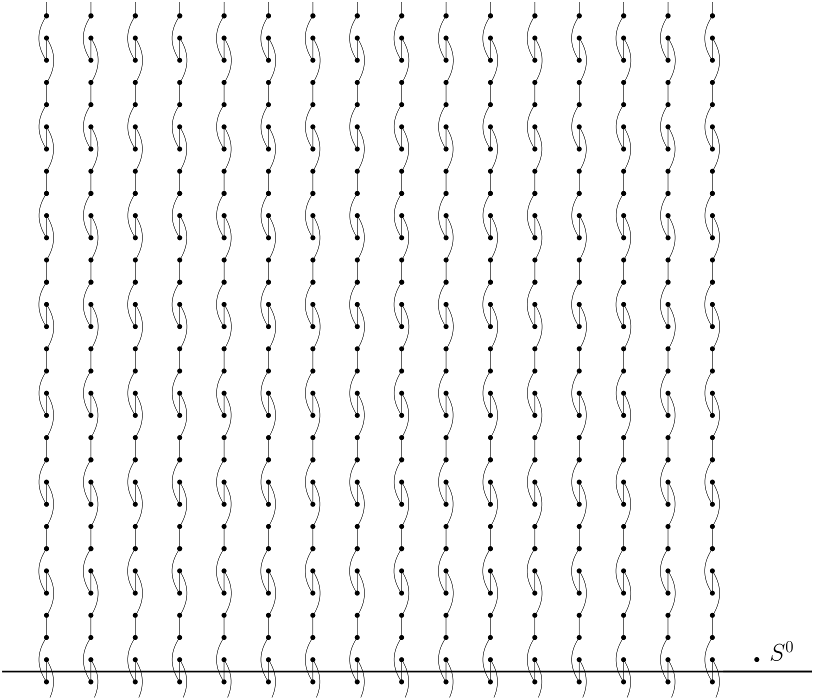
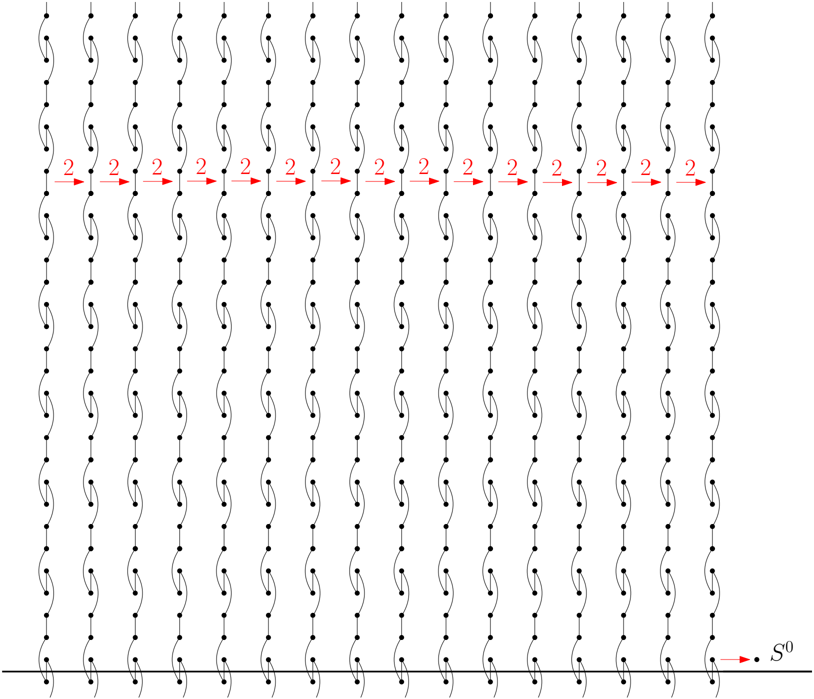
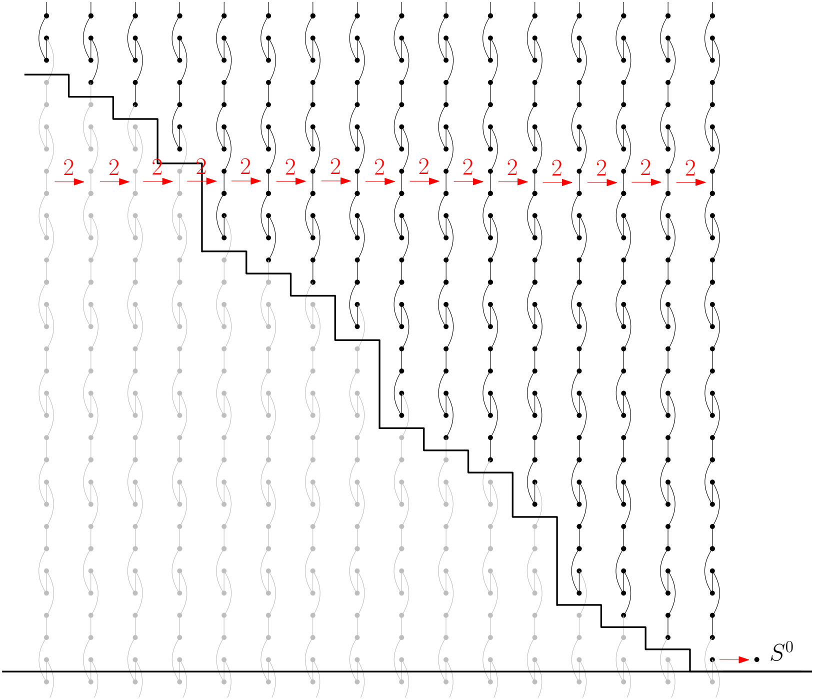
- Bruner–Greenlees: It is $|M(2^q)|\sigma$.
  $M(-)$: classical Mahowald invariant


S^0



2
S^0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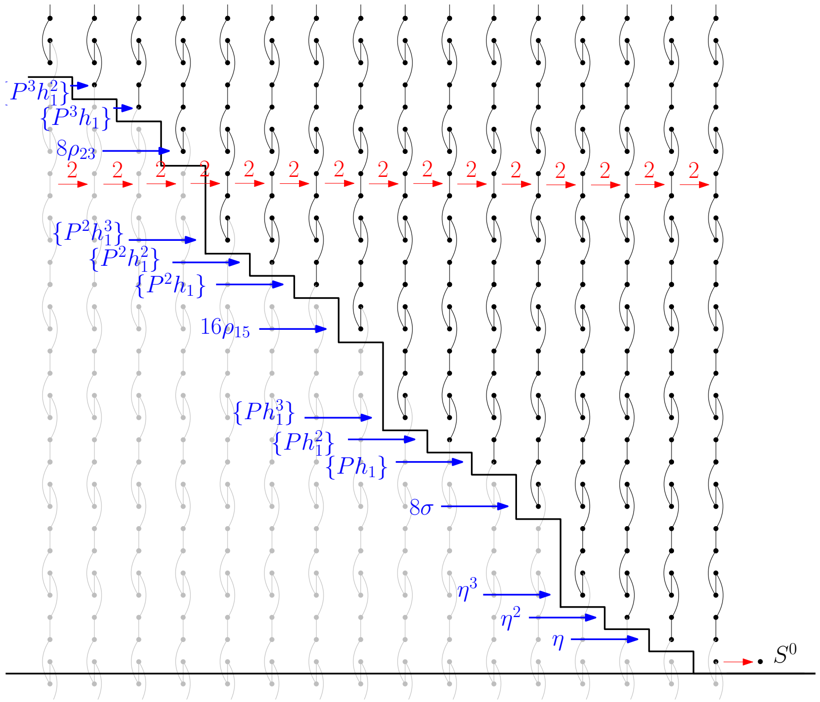
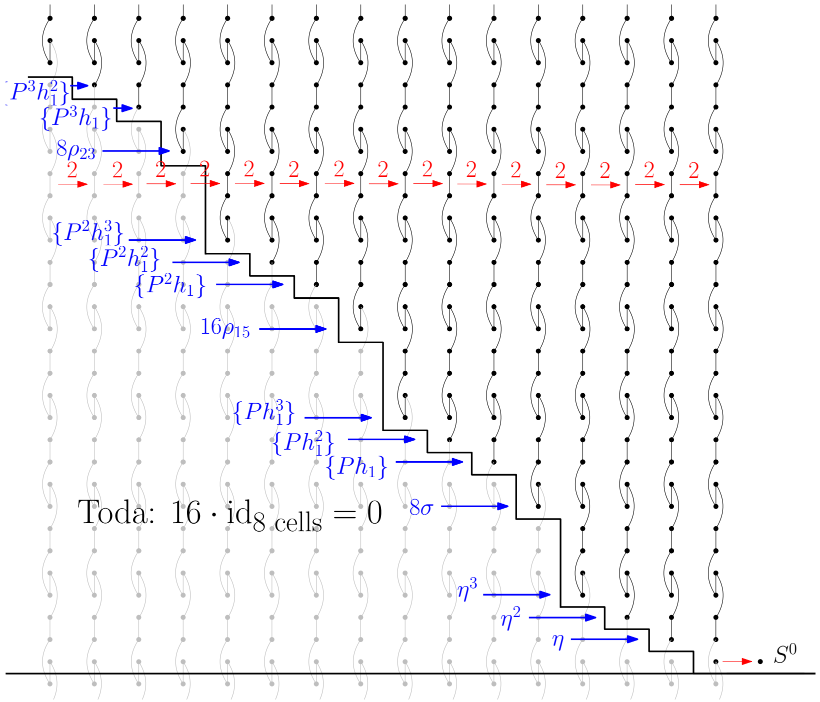
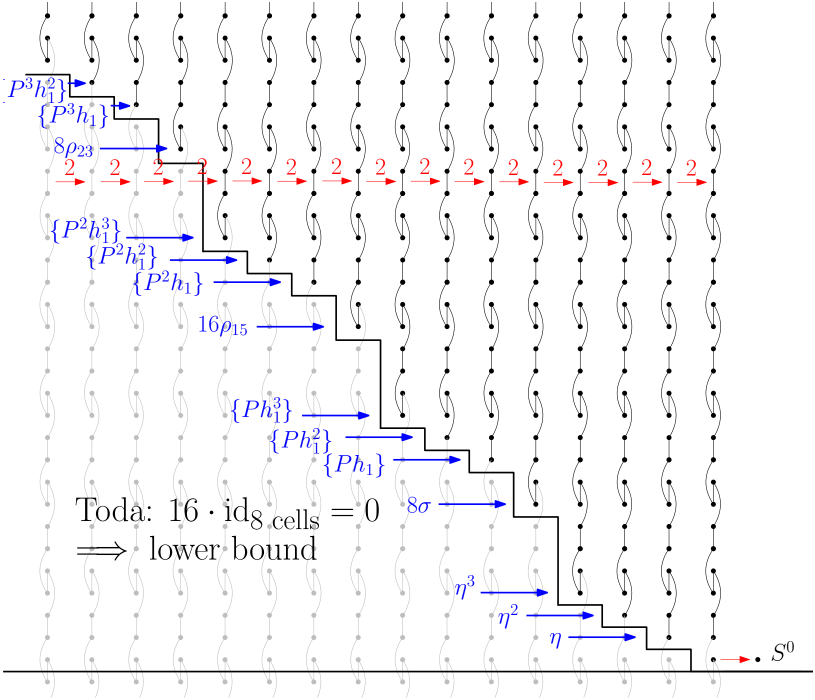


{P^3h_1^2}
{P^3h_1}
8ρ_23
2
{P^2h_1^3}
{P^2h_1^2}
{P^2h_1}
16ρ_15
{Ph_1^3}
{Ph_1^2}
{Ph_1}
8σ
η^3
η^2
η
S^0



$\{P^3h_1^2\}$
$\{P^3h_1\}$
$8\rho_{23}$
2
$\{P^2h_1^3\}$
$\{P^2h_1^2\}$
$\{P^2h_1\}$
$16\rho_{15}$
$\{Ph_1^3\}$
$\{Ph_1^2\}$
$\{Ph_1\}$
8σ
Toda: $16 \cdot \mathrm{id}_{8\ \mathrm{cells}} = 0$
η^3
η^2
η
S^0



$\{P^3h_1^2\}$
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$8\rho_{23}$
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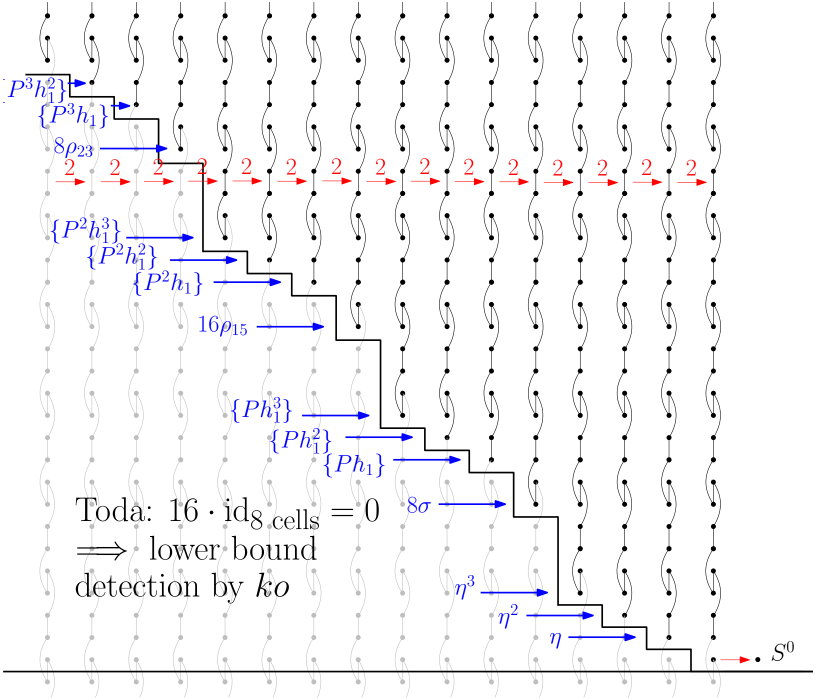
$\{Ph_1^2\}$
$\{Ph_1\}$
8σ
η^3
η^2
η
S^0
Toda: $16 \cdot \mathrm{id}_{8 \text{ cells}} = 0$
$\implies$ lower bound



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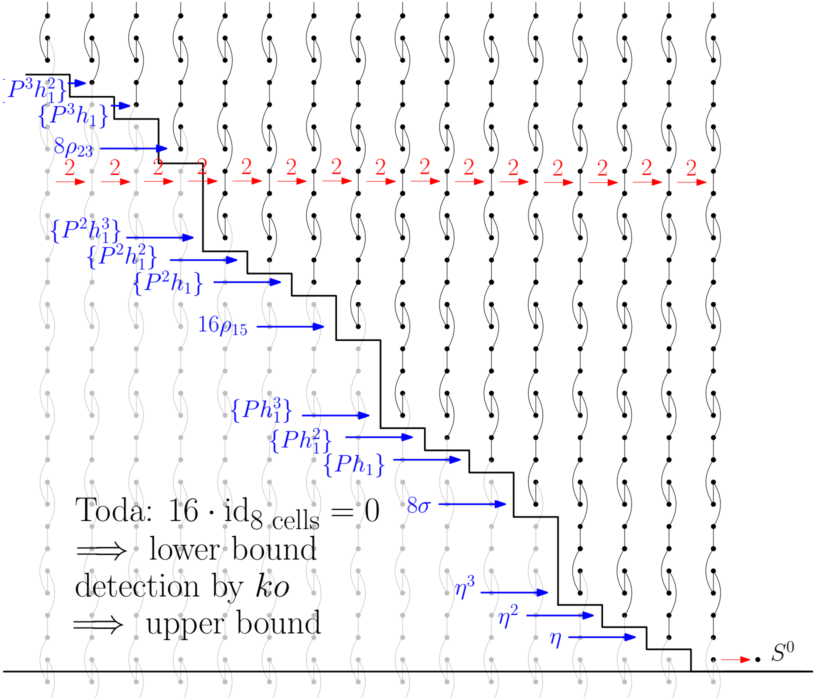
$\{P^3h_1\}$
$8\rho_{23}$
2
$\{P^2h_1^3\}$
$\{P^2h_1^2\}$
$\{P^2h_1\}$
$16\rho_{15}$
$\{Ph_1^3\}$
$\{Ph_1^2\}$
$\{Ph_1\}$
8σ
η^3
η^2
η
S^0
Toda: $16 \cdot \mathrm{id}_{8\ \mathrm{cells}} = 0$
$\Longrightarrow$ lower bound
detection by ko



{P^3h_1^2}
{P^3h_1}
8ρ_23
2
{P^2h_1^3}
{P^2h_1^2}
{P^2h_1}
16ρ_15
{Ph_1^3}
{Ph_1^2}
{Ph_1}
8σ
η^3
η^2
η
S^0
Toda: 16 · id_8 cells = 0
⟹ lower bound
detection by ko
⟹ upper bound

Pin(2)-equivariant to non-equivariant

- C_2-action on $BS^1 = \mathbb{C}P^\infty$:
  $(z_1, z_2, z_3, z_4, \ldots, z_{2n-1}, z_{2n}) \longmapsto$
  $(-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3, \ldots, -\bar{z}_{2n}, \bar{z}_{2n-1})$

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- $B\,\mathrm{Pin}(2) = BS^1/C_2$-action

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- $B\,\mathrm{Pin}(2) = BS^1/C_2$-action
- λ: line bundle associated to the principal bundle
  $C_2 \hookrightarrow BS^1 \longrightarrow B\,\mathrm{Pin}(2)$

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Pin(2)-equivariant to non-equivariant

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  - inclusion of bundles $m\lambda \hookrightarrow (m+1)\lambda$
    $\Longrightarrow X(m+1) \longrightarrow X(m)$

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    $\implies X(m+1) \longrightarrow X(m)$
    $\implies X(m+1) \longrightarrow X(m) \longrightarrow \Sigma^{-m}\mathbb{C}P^\infty$

Pin(2)-equivariant to non-equivariant

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- fiber bundle $\mathbb{R}P^2 \hookrightarrow B\,\mathrm{Pin}(2) \longrightarrow \mathbb{H}P^\infty$

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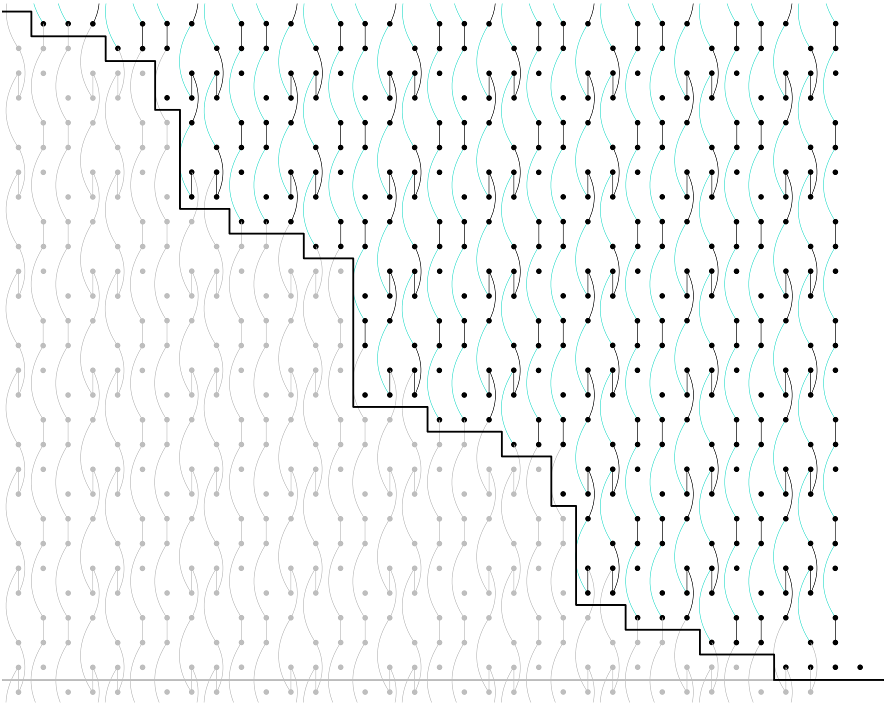
- fiber bundle $\mathbb{R}P^2 \hookrightarrow B\,\mathrm{Pin}(2) \longrightarrow \mathbb{H}P^\infty$
  gives cell structures on $B\,\mathrm{Pin}(2)$ and $X(m)$.



Mahowald line

Consider the diagram

$$\begin{array}{ccc} S^{p\mathbb{H}} & & \\ \uparrow{\scriptstyle a_{\mathbb{H}}^{p}} & \overset{\exists}{\searrow} & \\ S^0 & \xrightarrow{a_{\widetilde{\mathbb{R}}}^{q}} & S^{q\widetilde{\mathbb{R}}} \end{array}$$

Mahowald line

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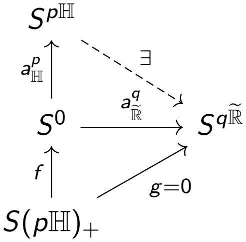

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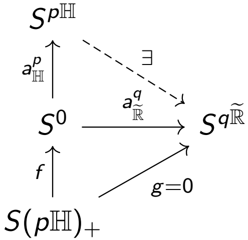

- g is zero $\iff$

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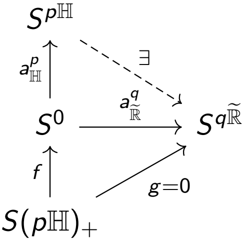
- g is zero $\iff$
  $S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+ \to S(p\mathbb{H})_+ \xrightarrow{f} S^0$ is zero

Mahowald line

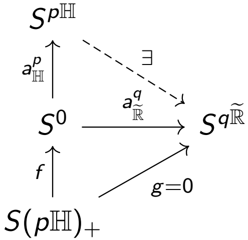
Consider the diagram



- g is zero $\iff$
  $S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+ \to S(p\mathbb{H})_+ \xrightarrow{f} S^0$ is zero
- $S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+$: Pin(2)-free

Mahowald line

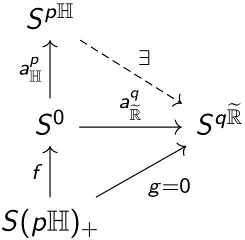
Consider the diagram



- g is zero $\iff$ $S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+ \to S(p\mathbb{H})_+ \xrightarrow{f} S^0$ is zero
- $S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+$: Pin(2)-free
- S^0: Pin(2) acts trivially

Mahowald line

Consider the diagram



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- $S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+$: Pin(2)-free
- S^0: Pin(2) acts trivially
- g is zero $\iff$ the nonequivariant map is zero

$$(S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+)_{h\,\mathrm{Pin}(2)} \longrightarrow (S(p\mathbb{H})_+)_{h\,\mathrm{Pin}(2)} \longrightarrow S^0$$

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- Short exact sequence
  $1 \longrightarrow S^1 \longrightarrow \mathrm{Pin}(2) \longrightarrow C_2 \longrightarrow 1$

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  $1 \longrightarrow S^1 \longrightarrow \mathrm{Pin}(2) \longrightarrow C_2 \longrightarrow 1$

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  $1 \longrightarrow S^1 \longrightarrow \mathrm{Pin}(2) \longrightarrow C_2 \longrightarrow 1$

$$\begin{aligned} (S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+)_{h\,\mathrm{Pin}(2)} &= \left((S^{-q\widetilde{\mathbb{R}}} \wedge S(p\mathbb{H})_+)_{hS^1} \right)_{hC_2} \\ &= \left(S^{-q\sigma} \wedge \mathbb{C}P_+^{2p-1} \right)_{hC_2} \end{aligned}$$

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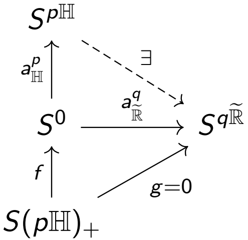
\end{aligned}
$$



Mahowald line

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$$\begin{aligned}
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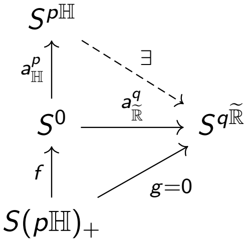
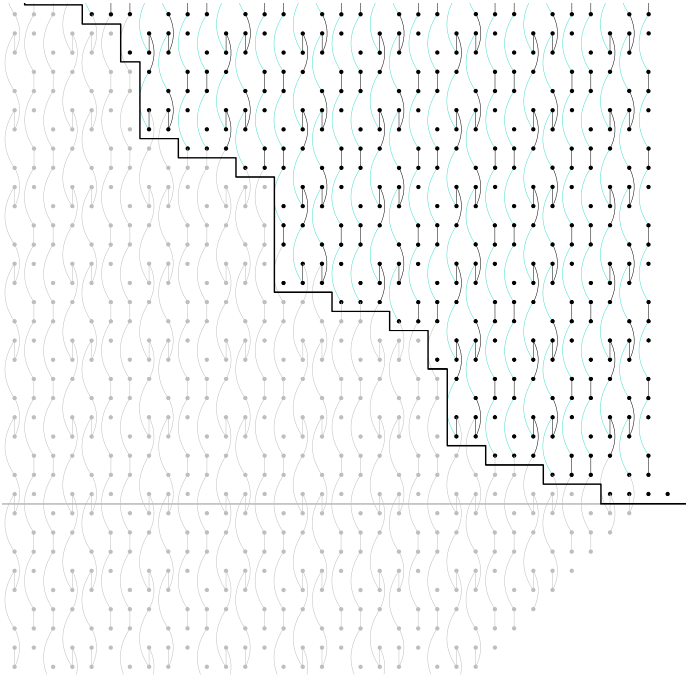
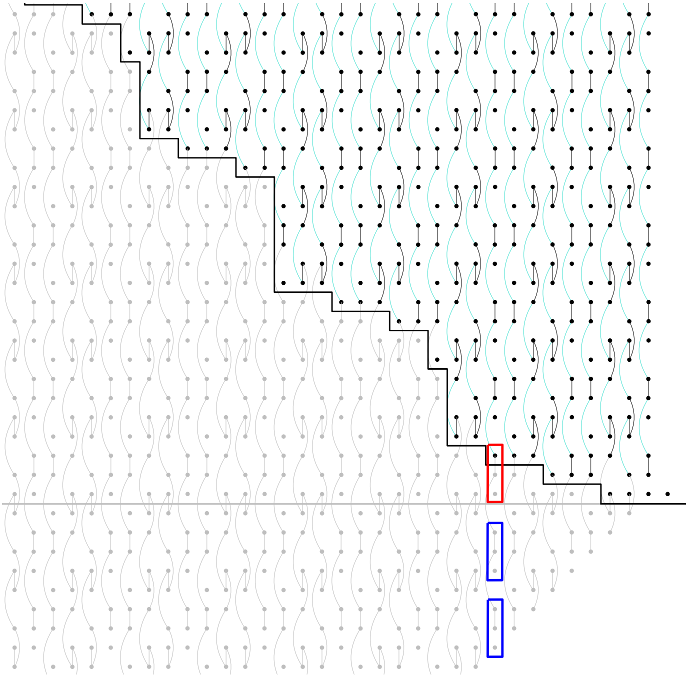
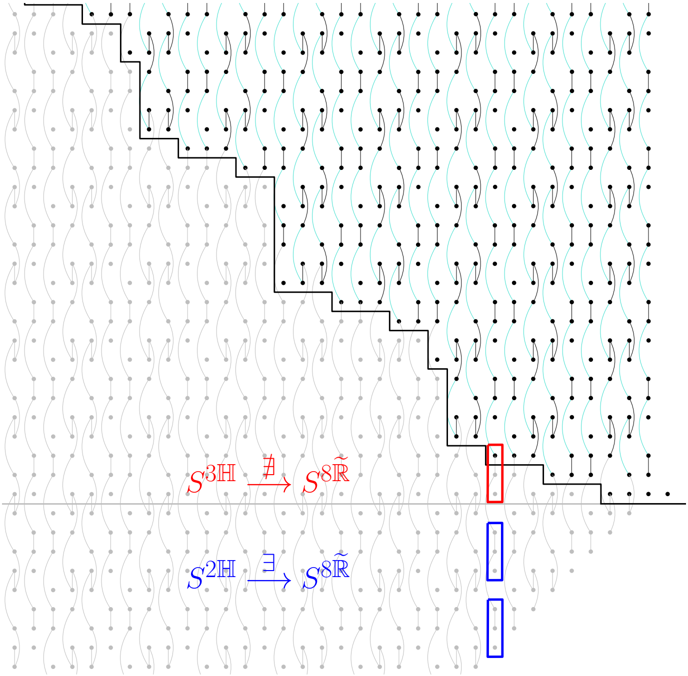


$\iff X(q)^{4p-2-q} \longrightarrow S^0$ is zero



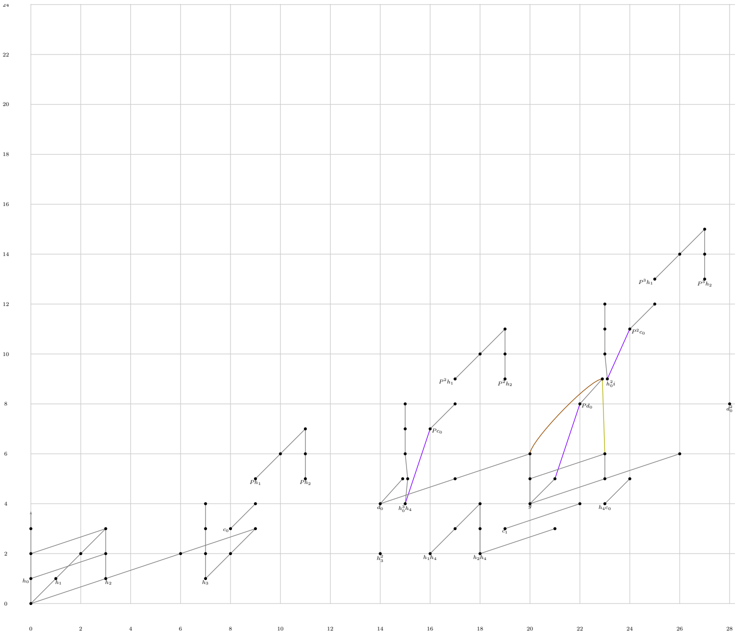




$S^{3\mathbb{H}} \xrightarrow{\nexists} S^{8\widetilde{\mathbb{R}}}$
$S^{2\mathbb{H}} \xrightarrow{\exists} S^{8\widetilde{\mathbb{R}}}$

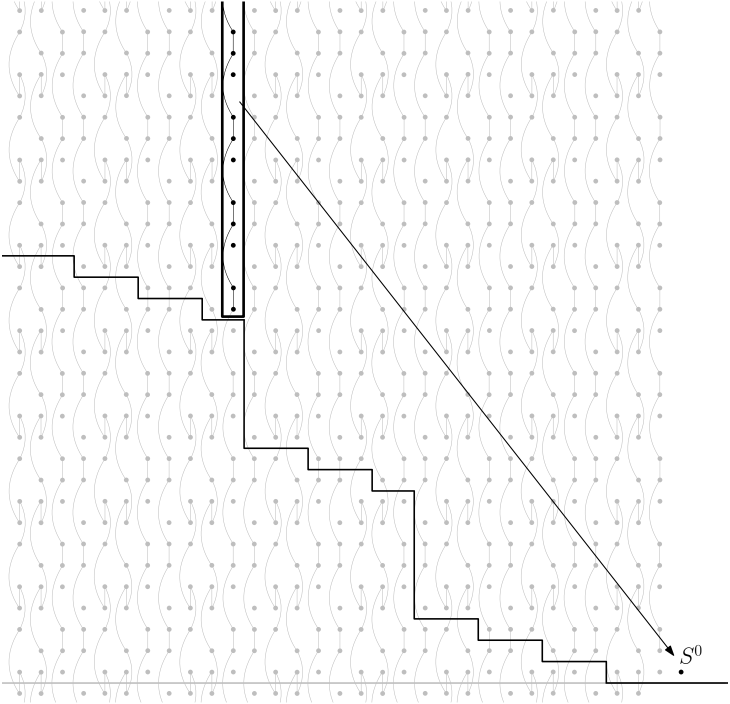
Lower bound

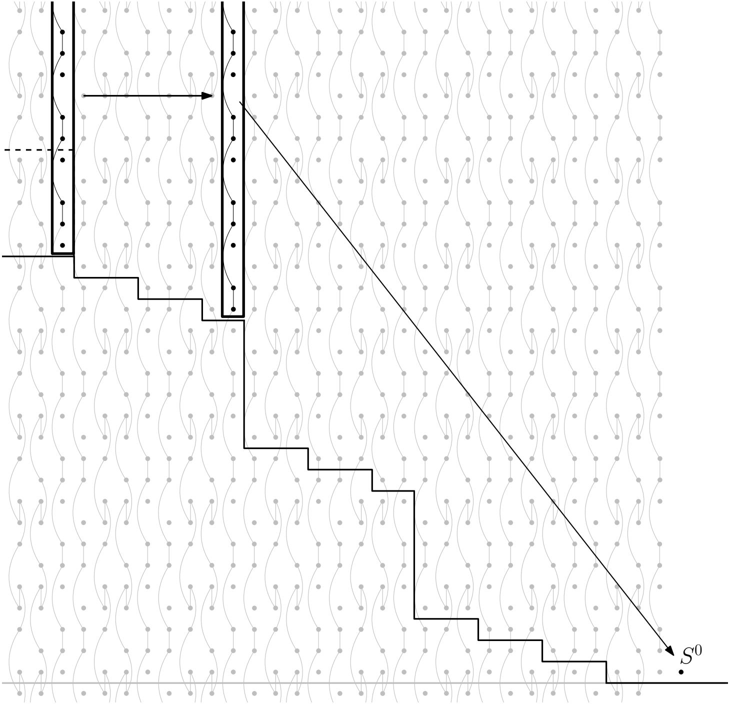
Classical Adams spectral sequence



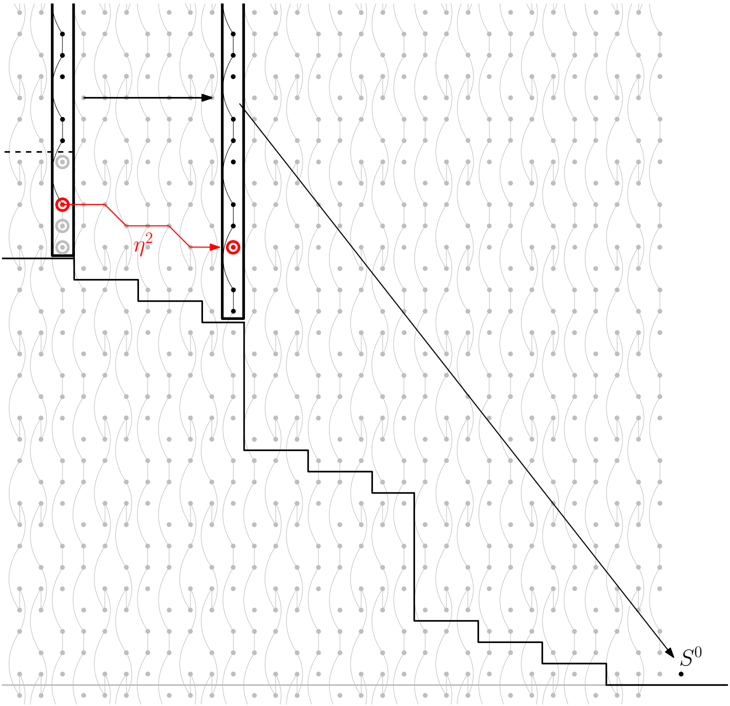
Some relations in $\pi_* S^0$

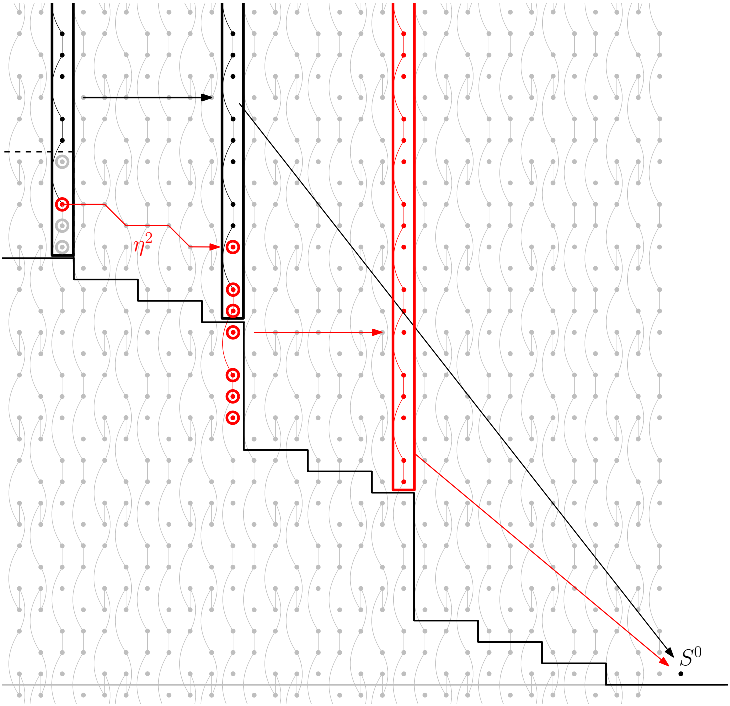
- $\pi_4 = 0$
- $\pi_5 = 0$
- $\pi_{12} = 0$
- $\pi_{13} = 0$
- $\eta \cdot \pi_6 = 0$
- $\pi_8 \cdot \eta^2 = 0$


S^0

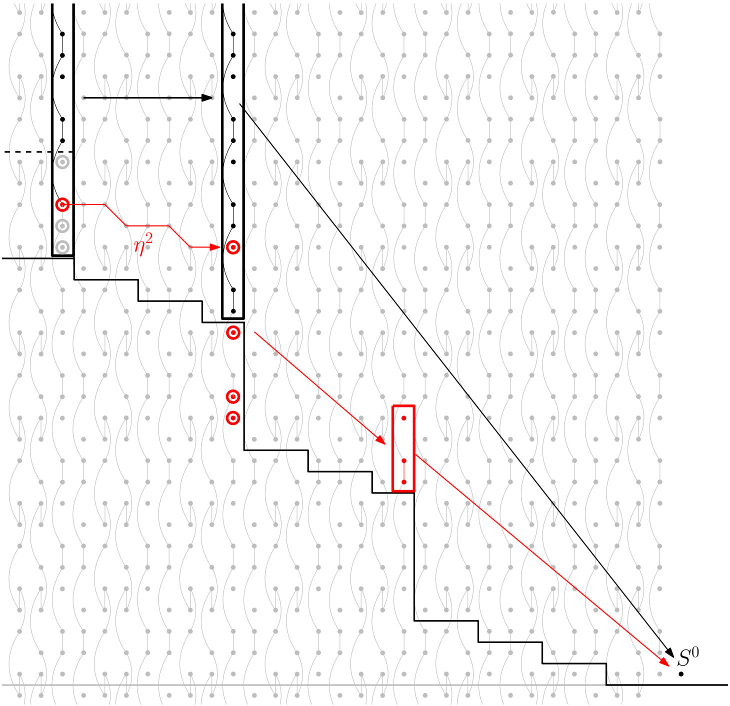


S^0

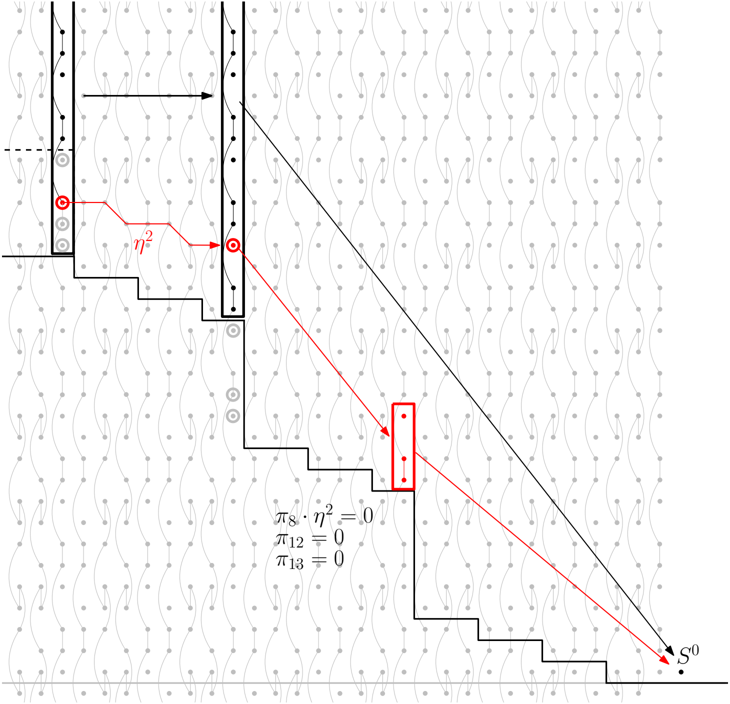

η^2
S^0



η^2
S^0

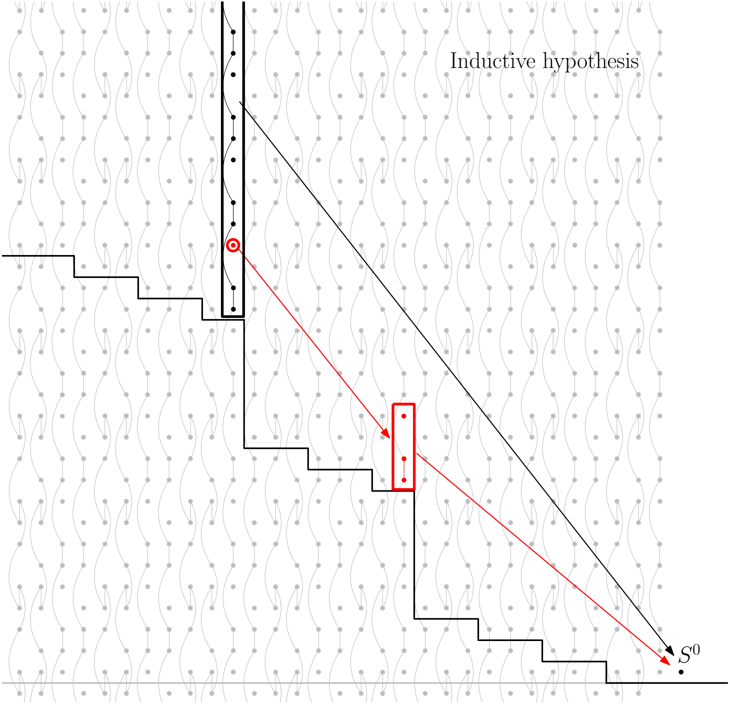


η^2
S^0



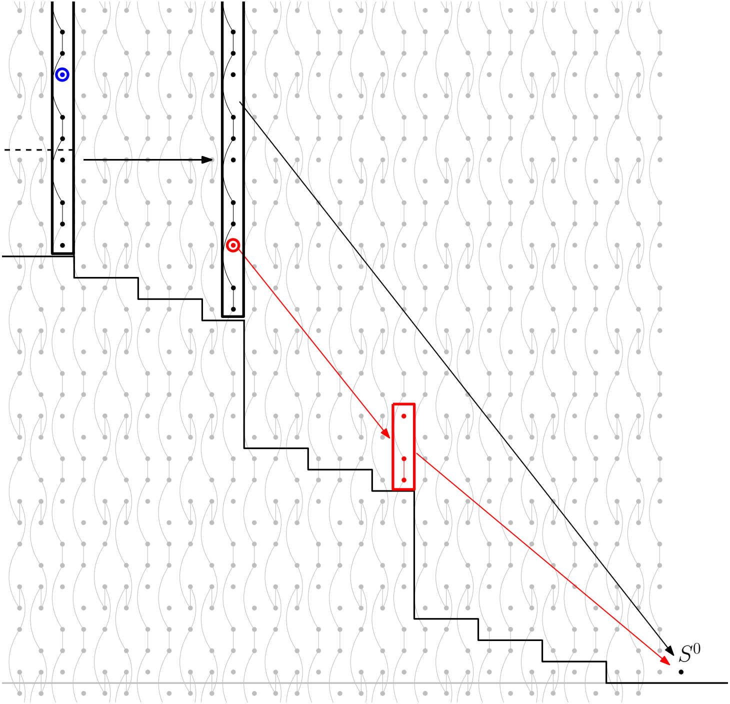
η^2
$\pi_8 \cdot \eta^2 = 0$
$\pi_{12} = 0$
$\pi_{13} = 0$
S^0

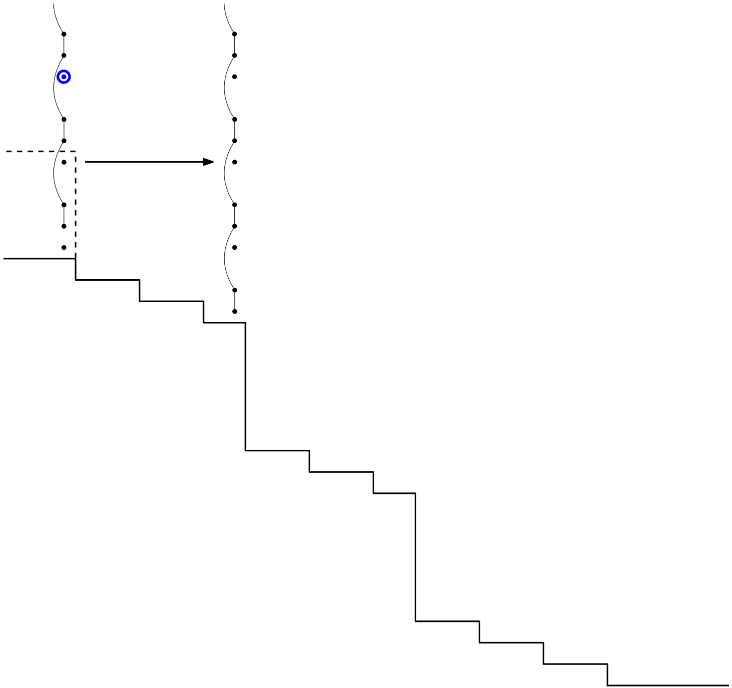
Now we start the induction

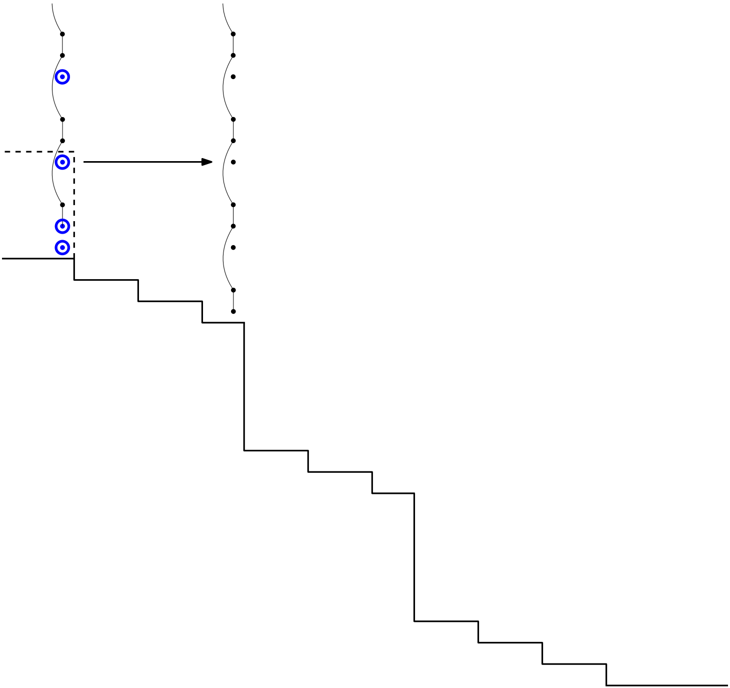


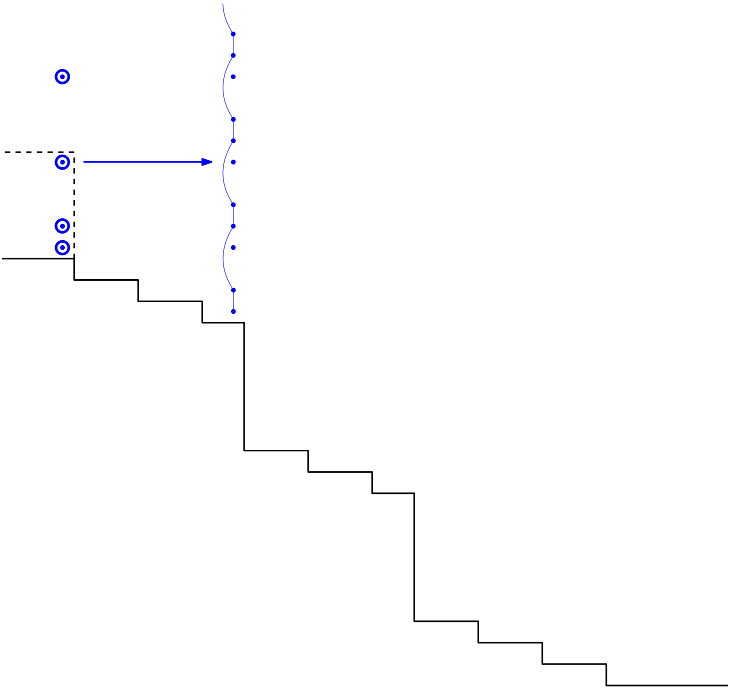
Inductive hypothesis
S^0

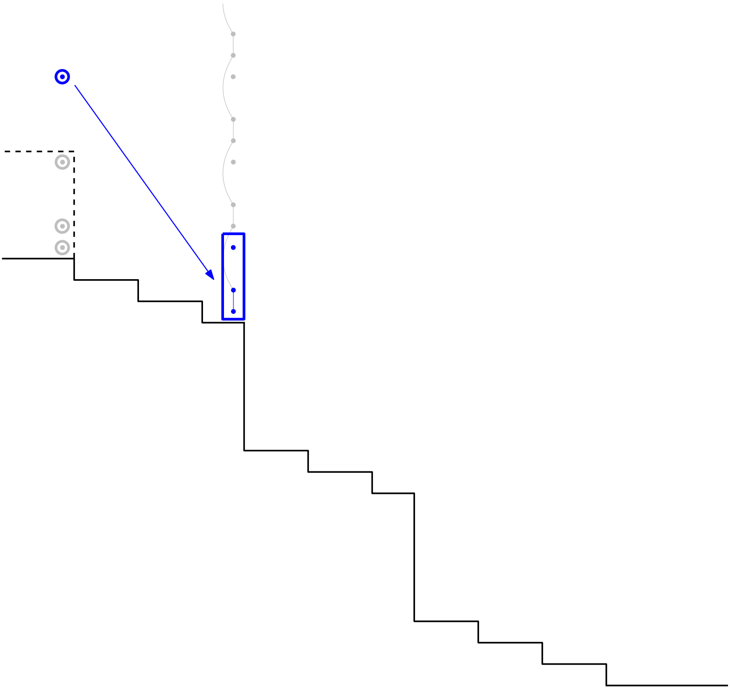

S^0

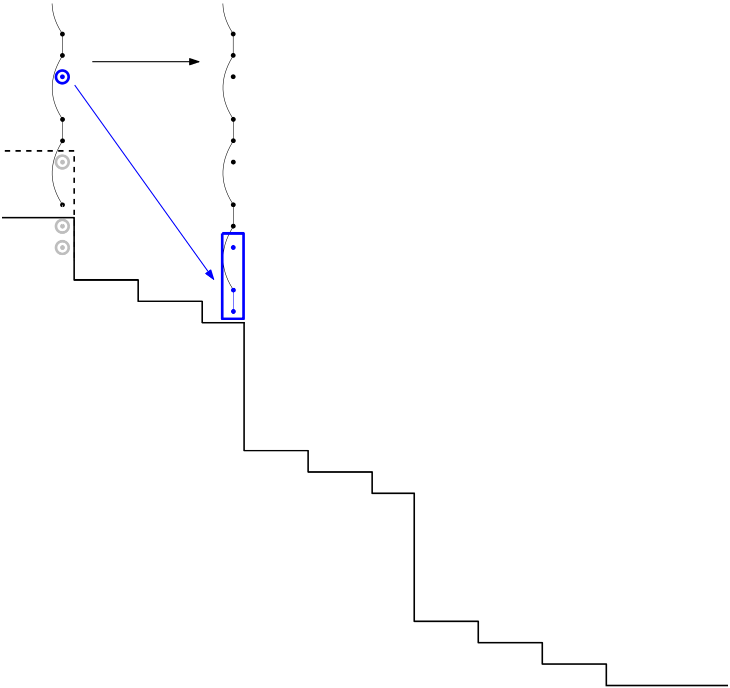

S^0

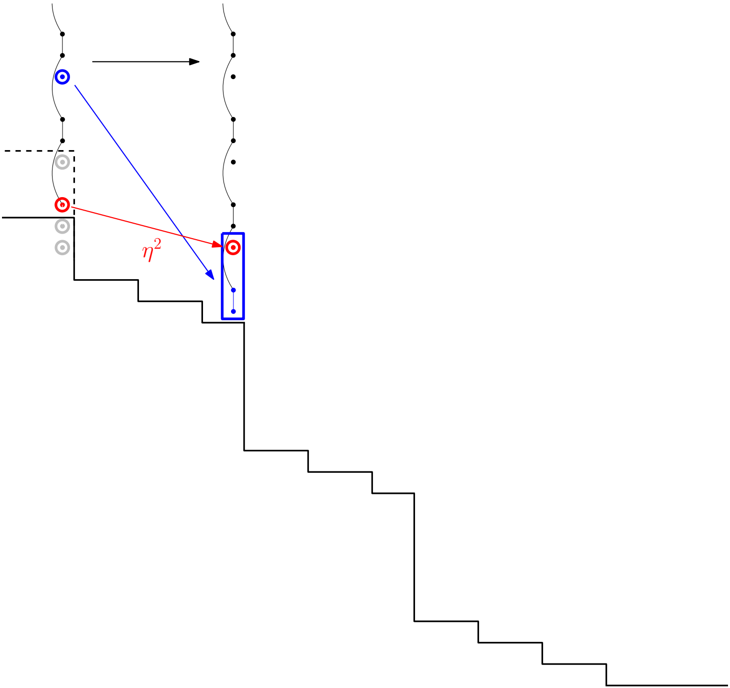


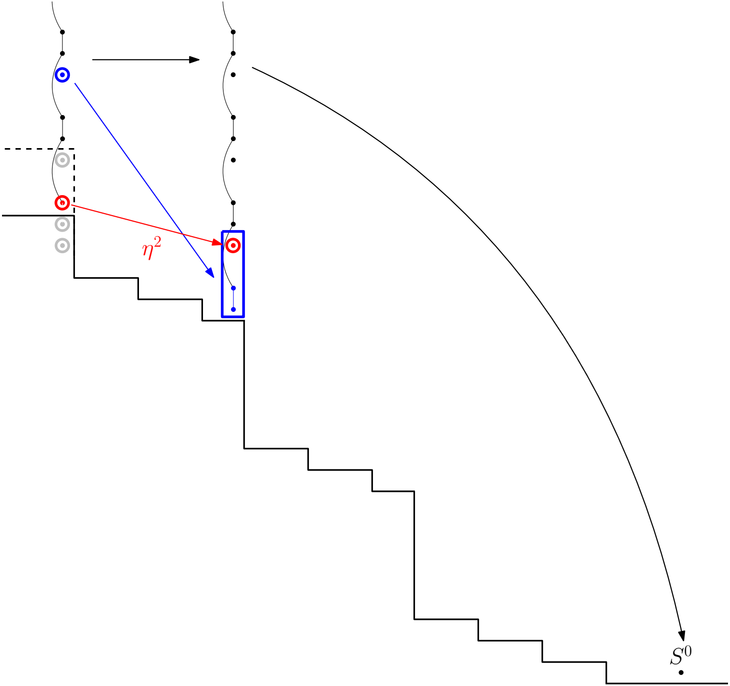




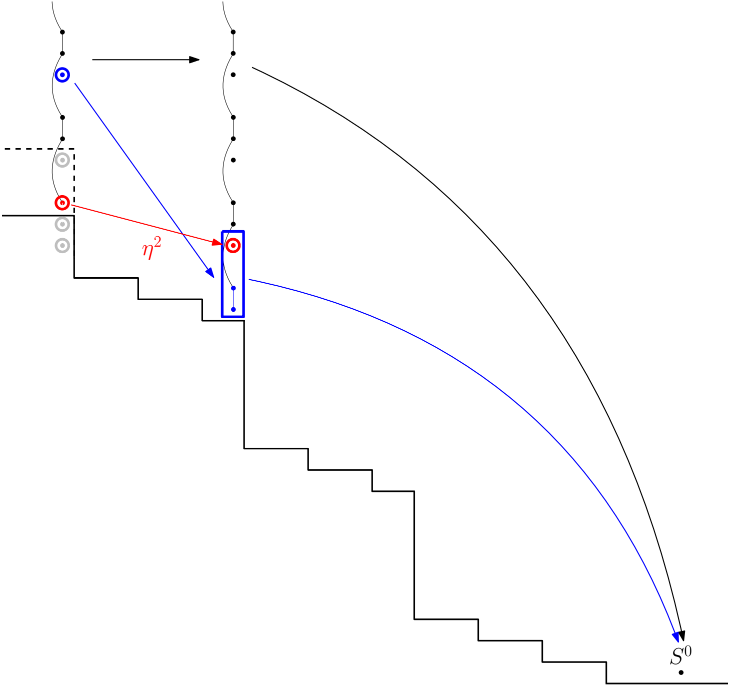




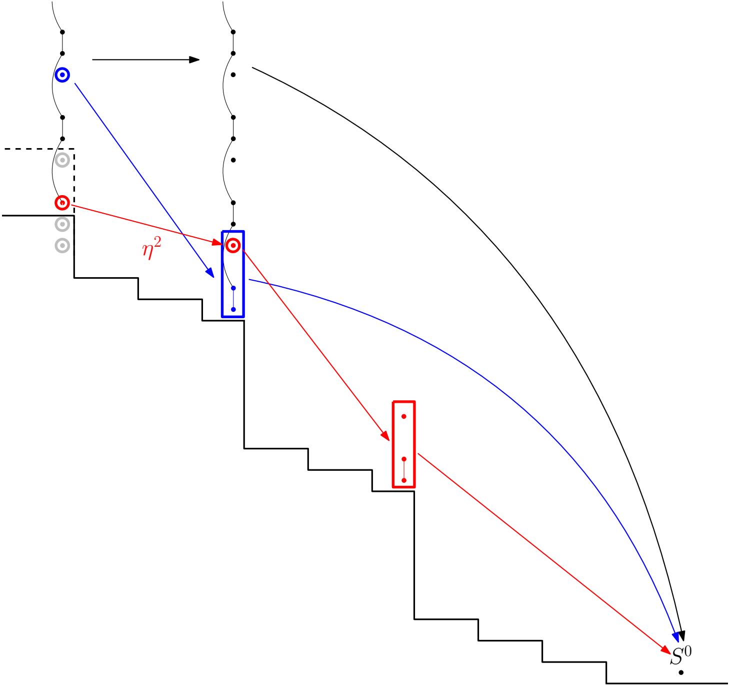

η^2



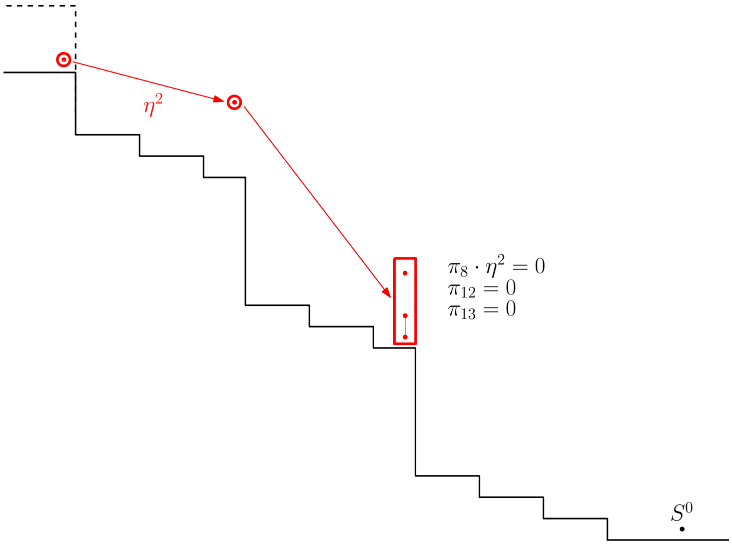
η^2
S^0



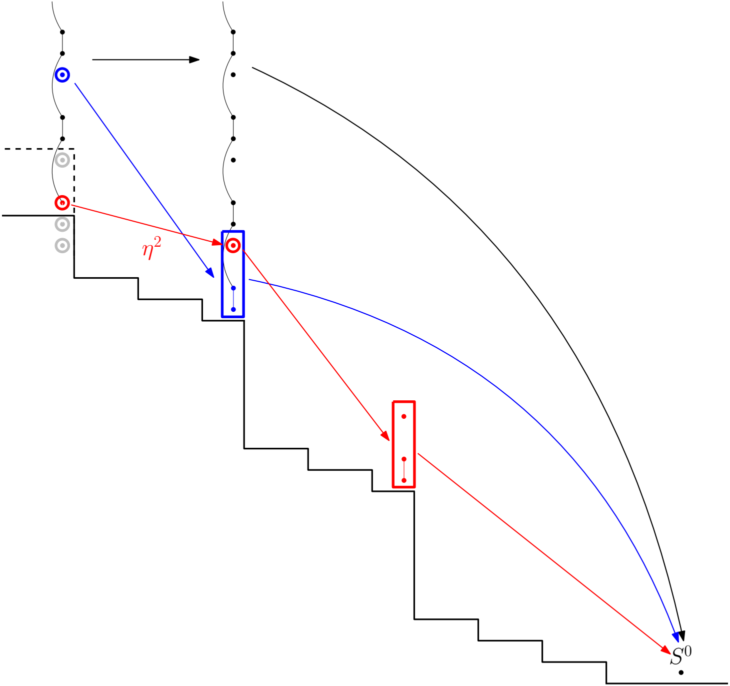
η^2
S^0

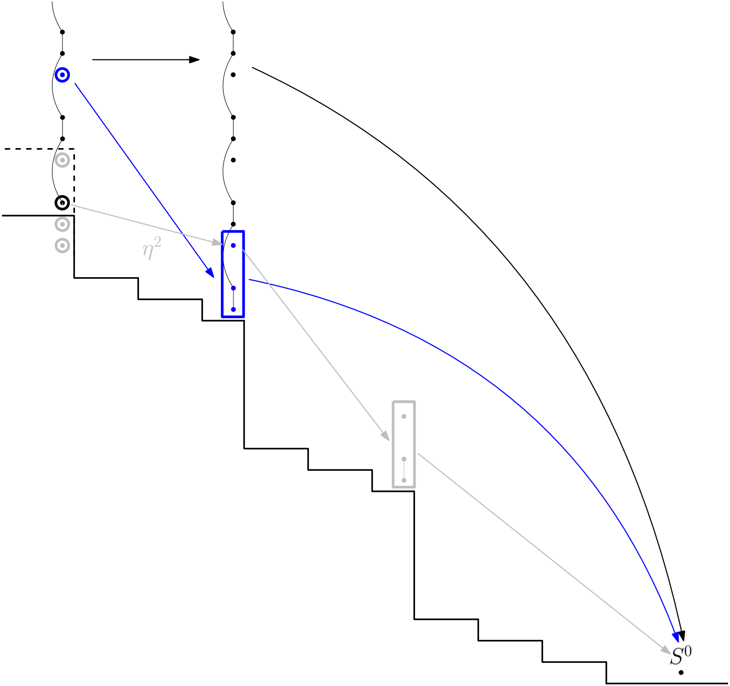


η^2
S^0

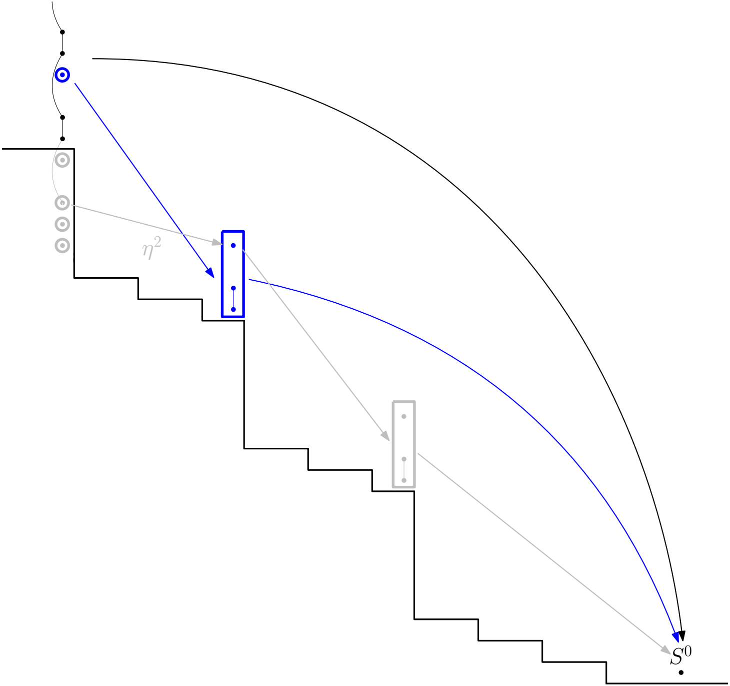


η^2
$\pi_8 \cdot \eta^2 = 0$
$\pi_{12} = 0$
$\pi_{13} = 0$
S^0

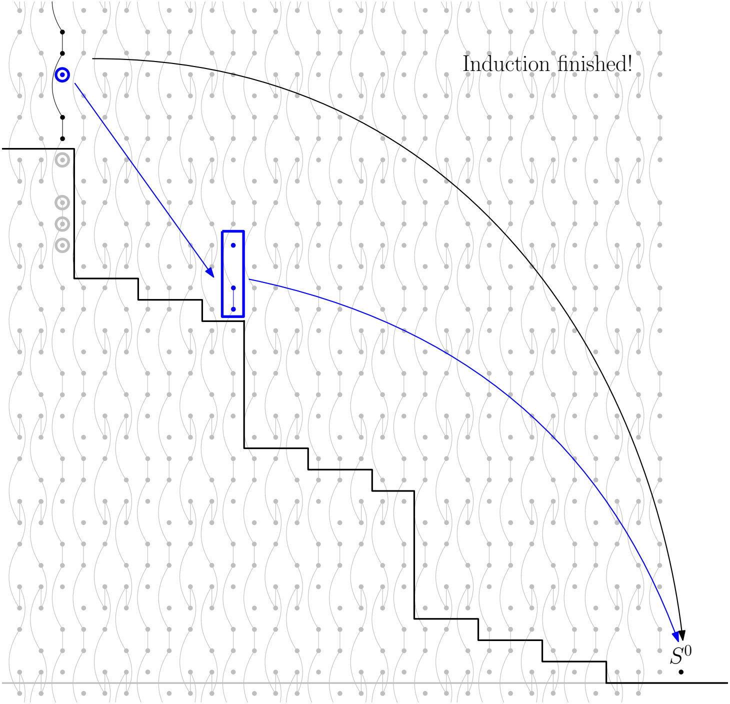

η^2
S^0



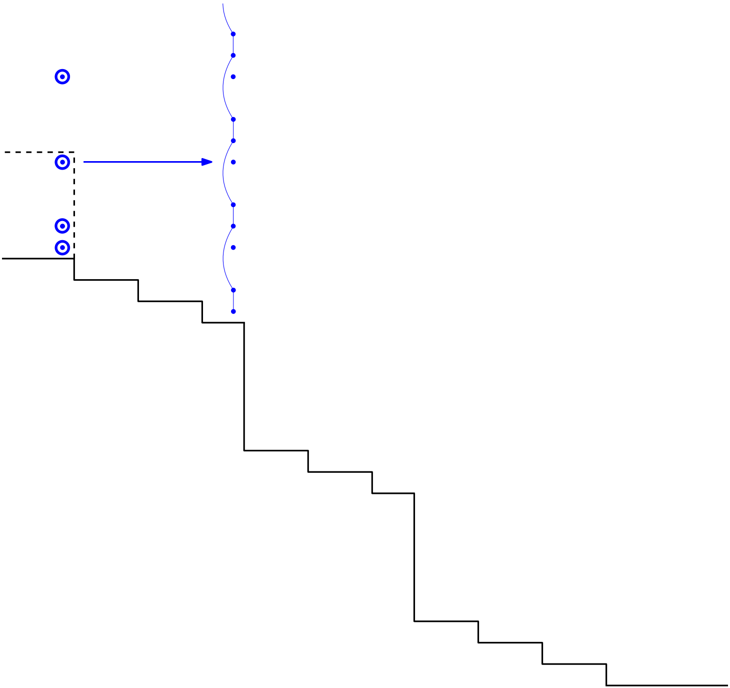
η^2
S^0

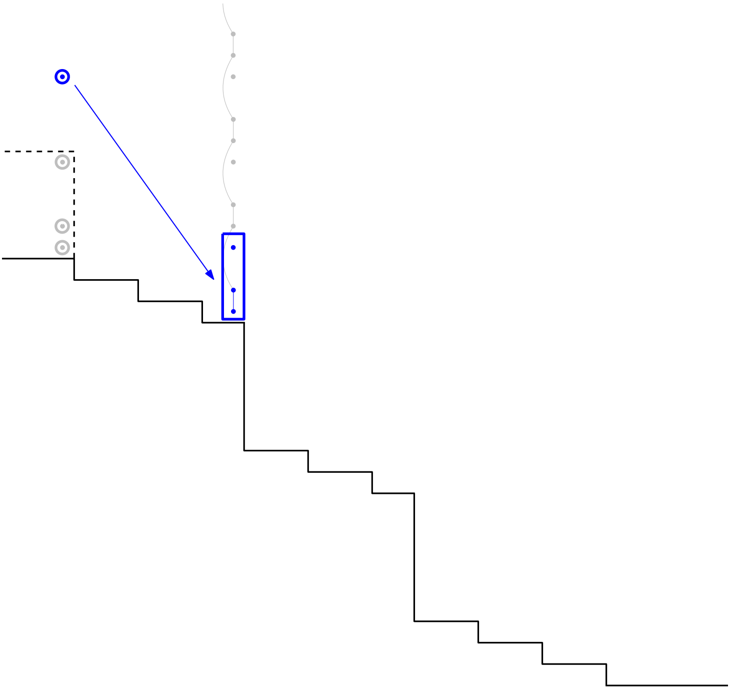


η^2
S^0

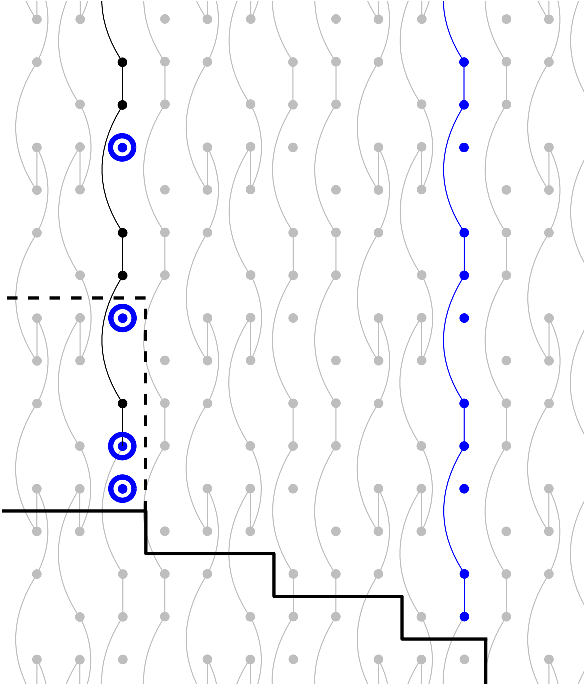

Induction finished!
S^0

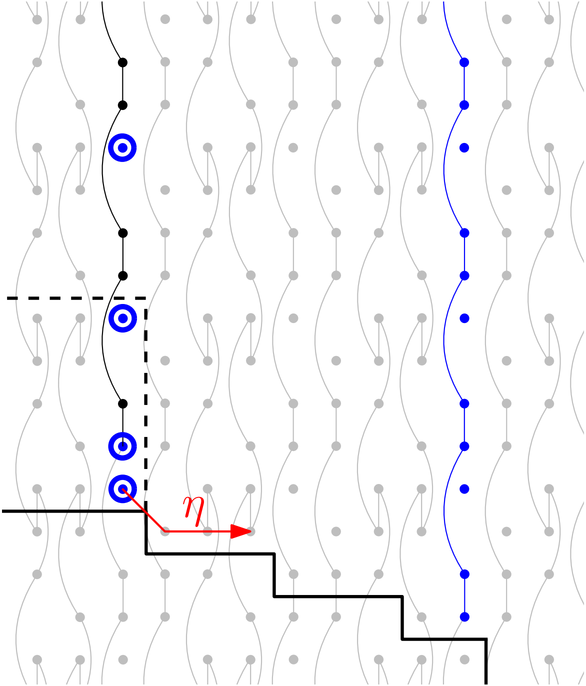
Intuition for a technical step

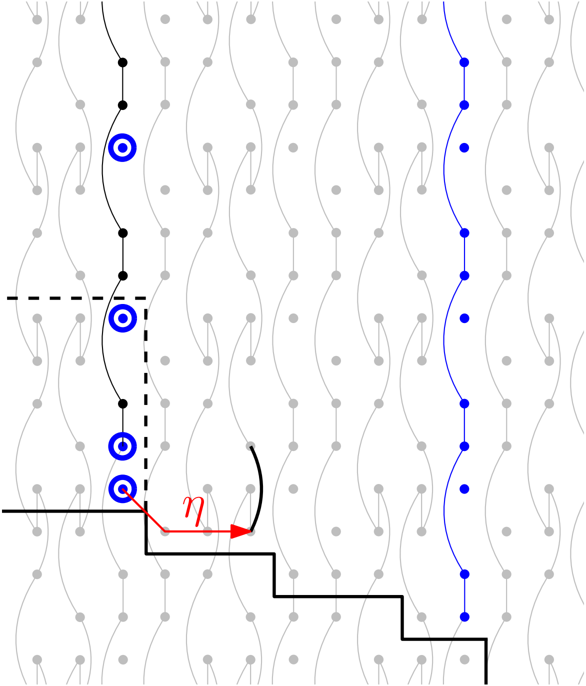




Another mini-movie




η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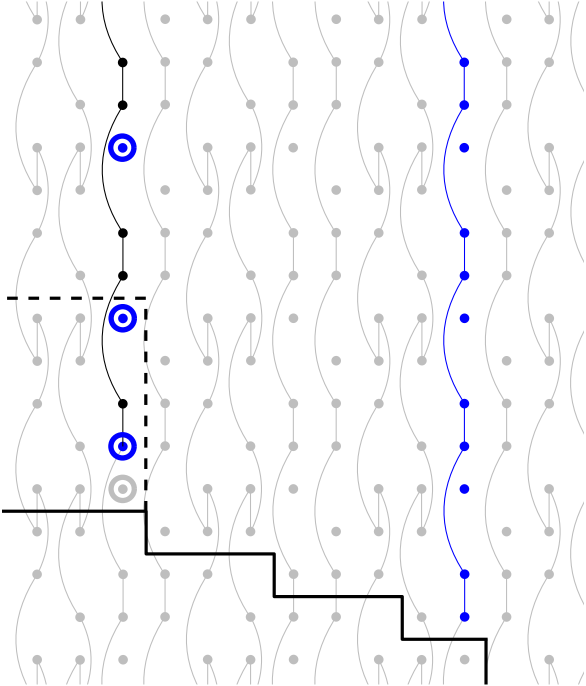
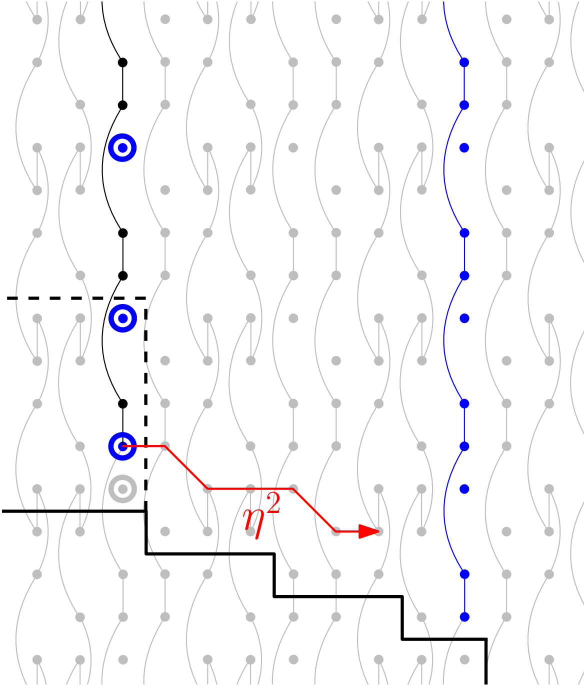
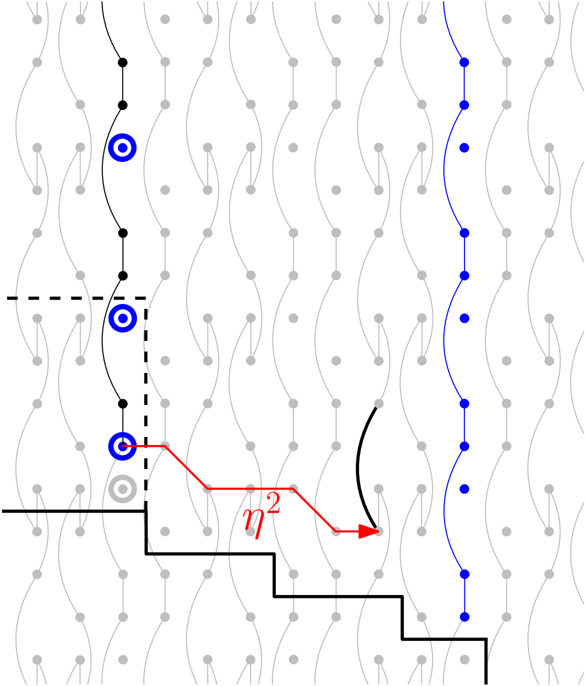
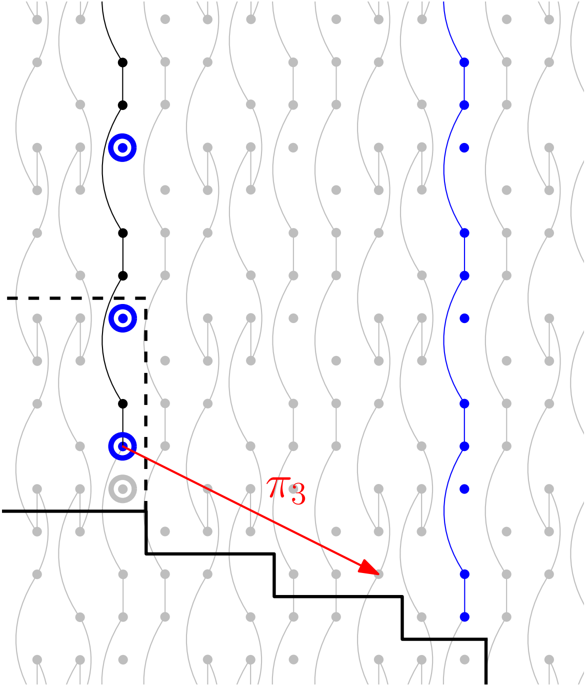
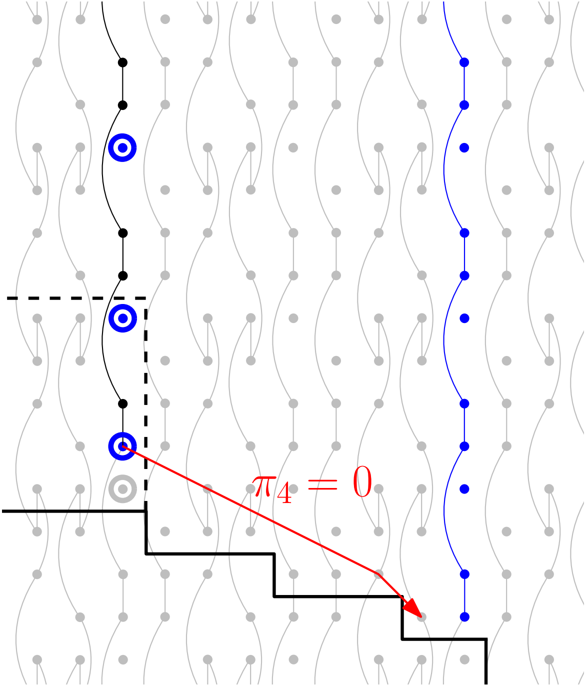



η^2


η^2



π_3



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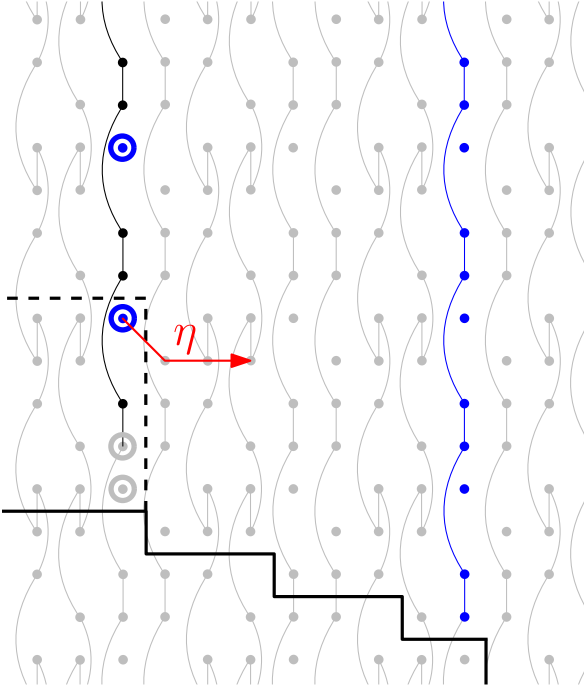
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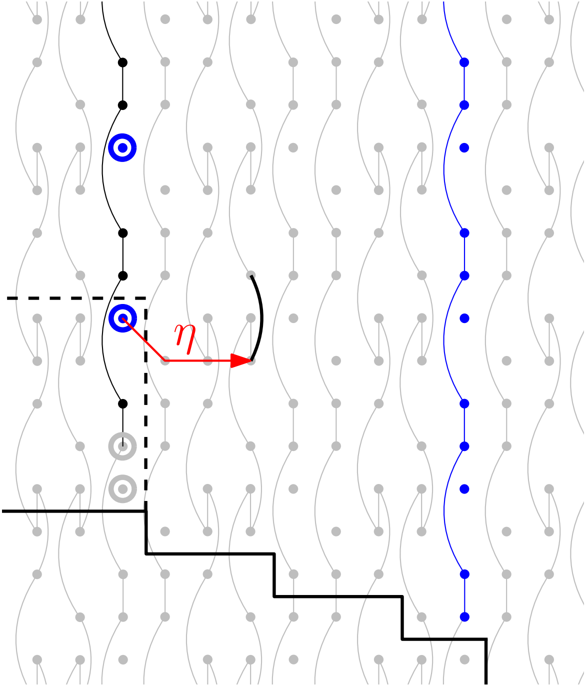
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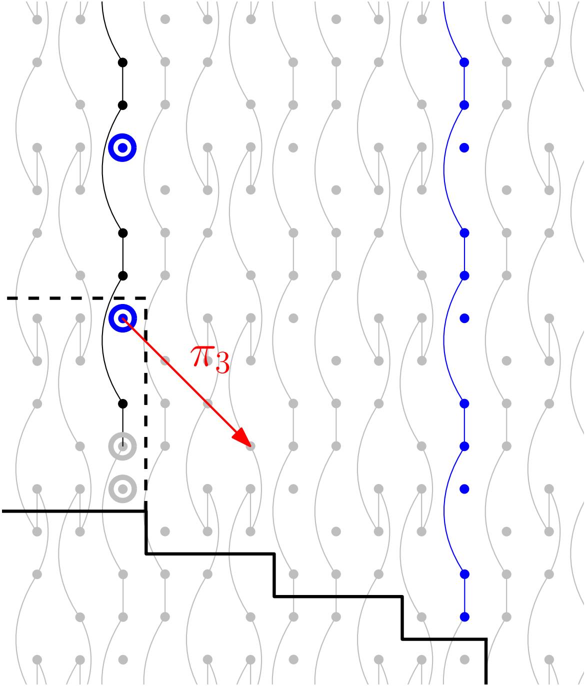
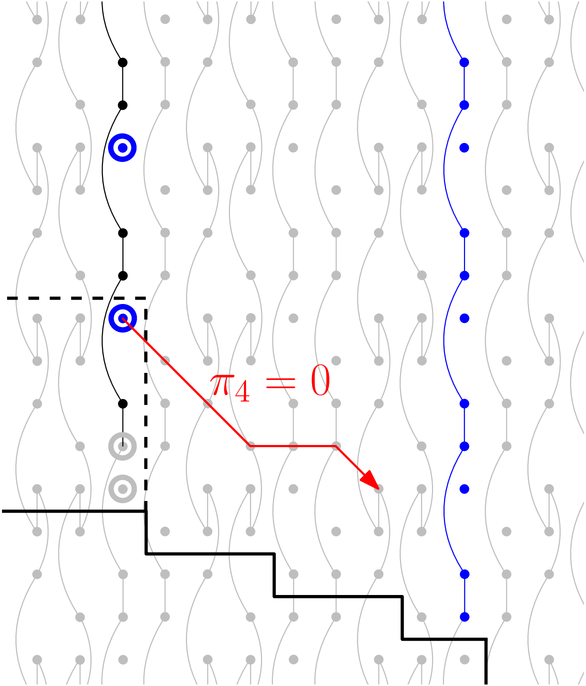

π_3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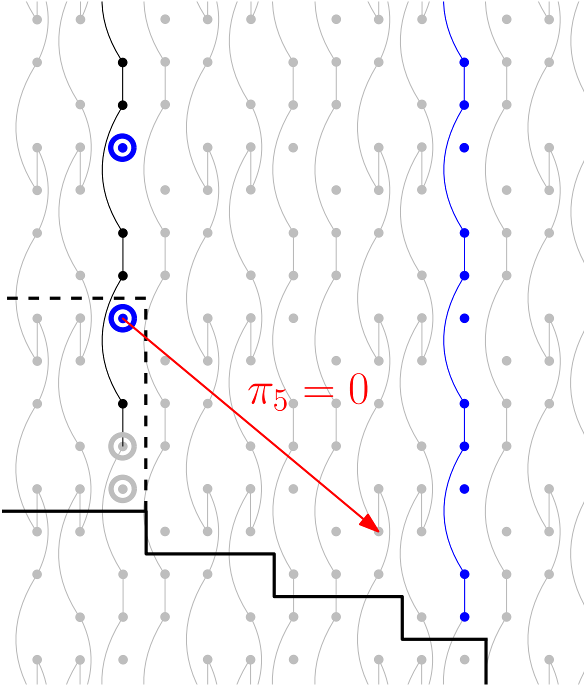
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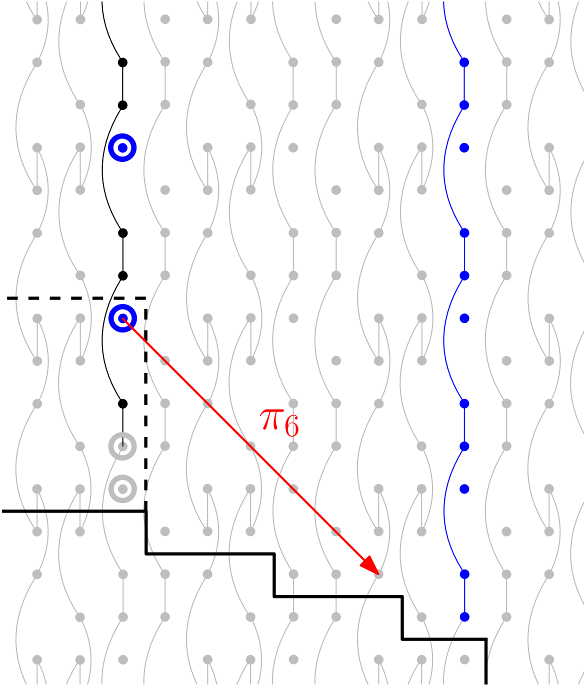
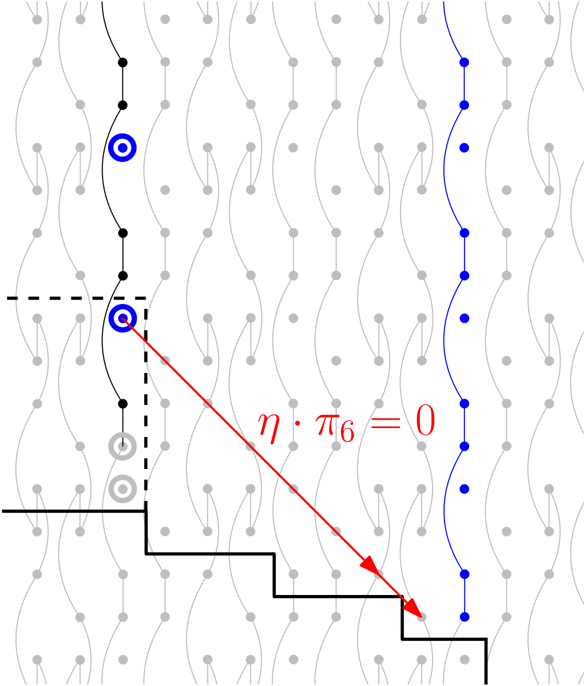
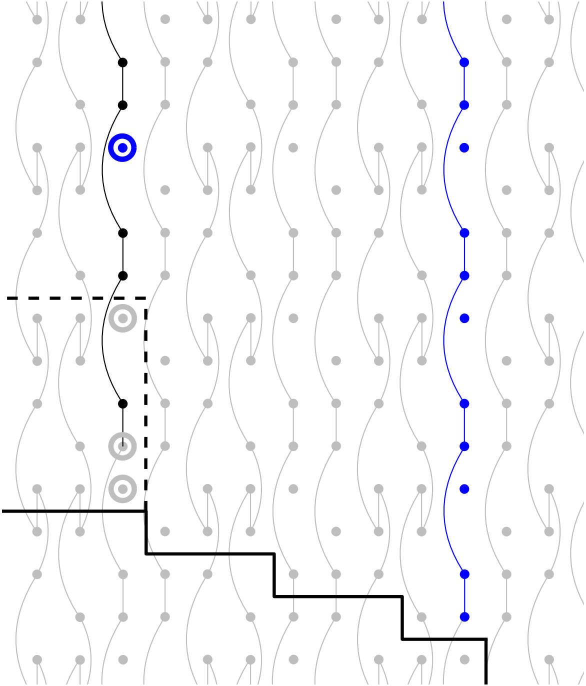
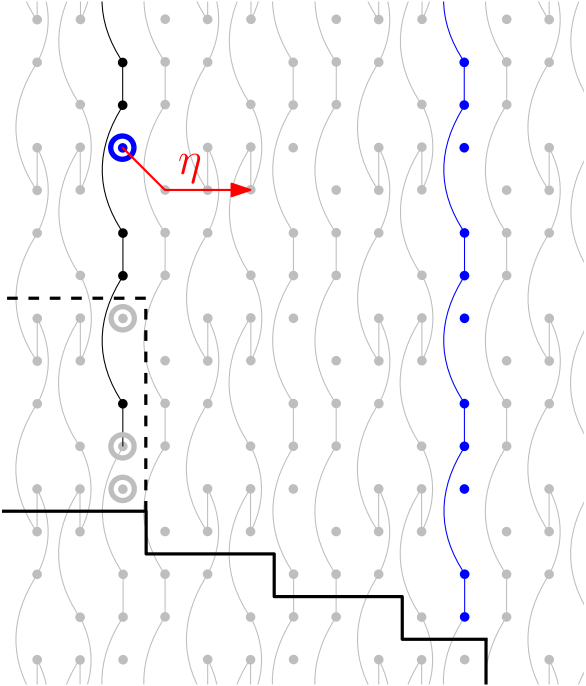
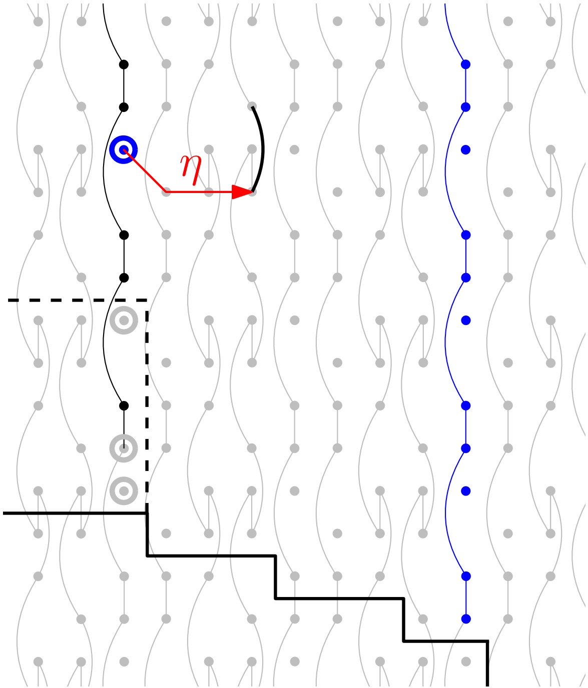

π_6


$\eta \cdot \pi_6 = 0$




η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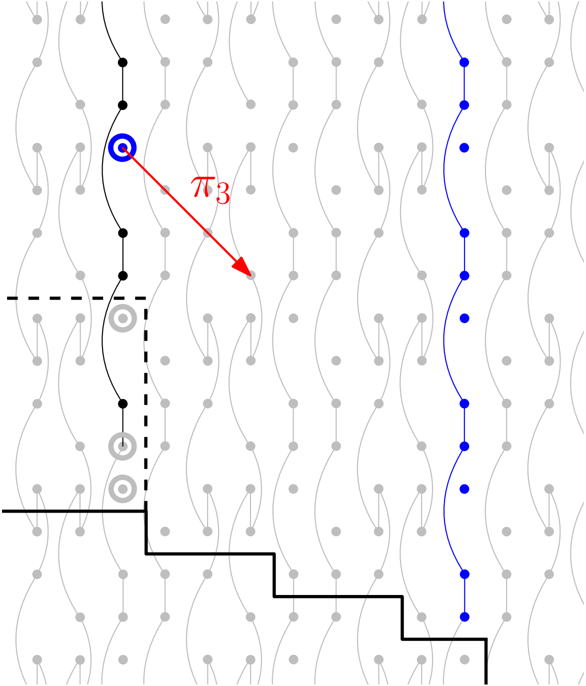
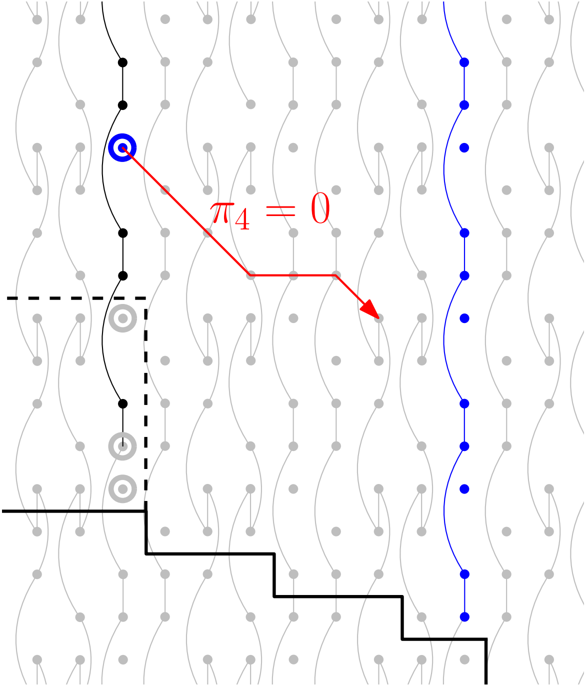

π_3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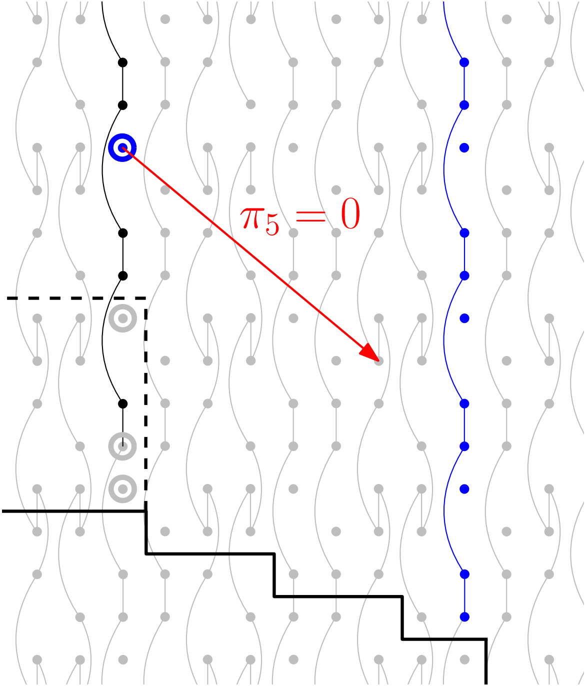
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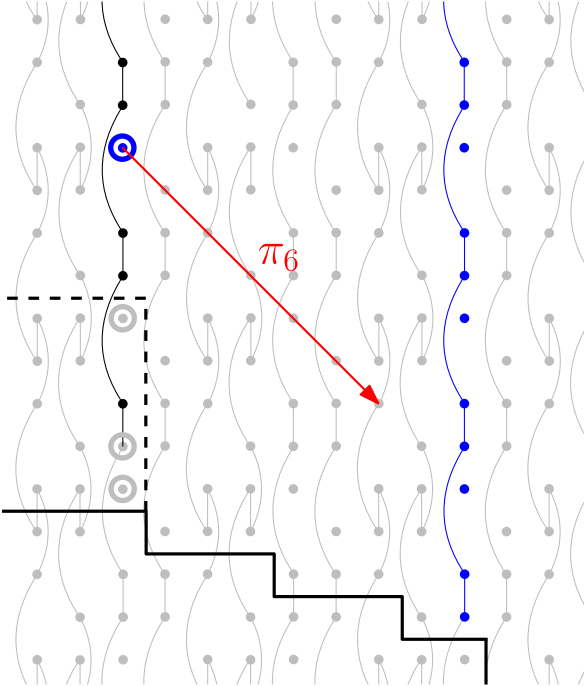
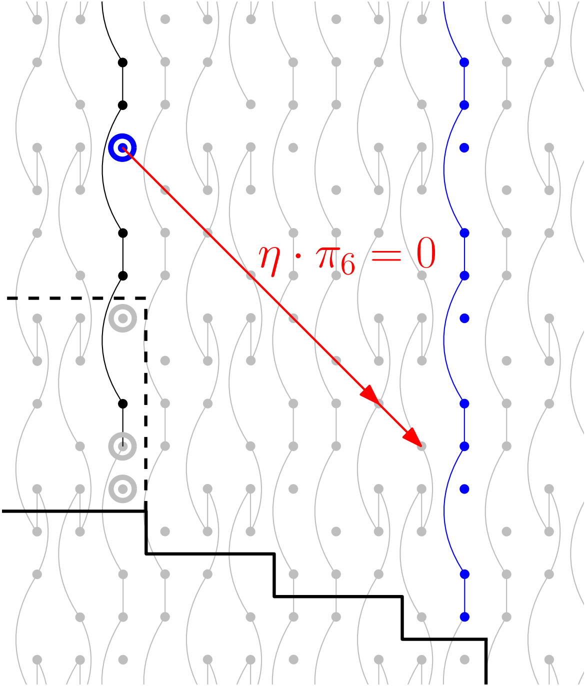


π_6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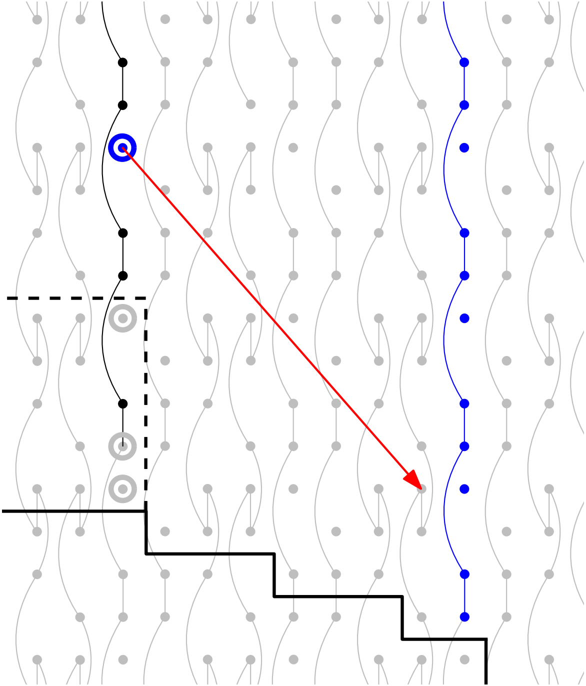

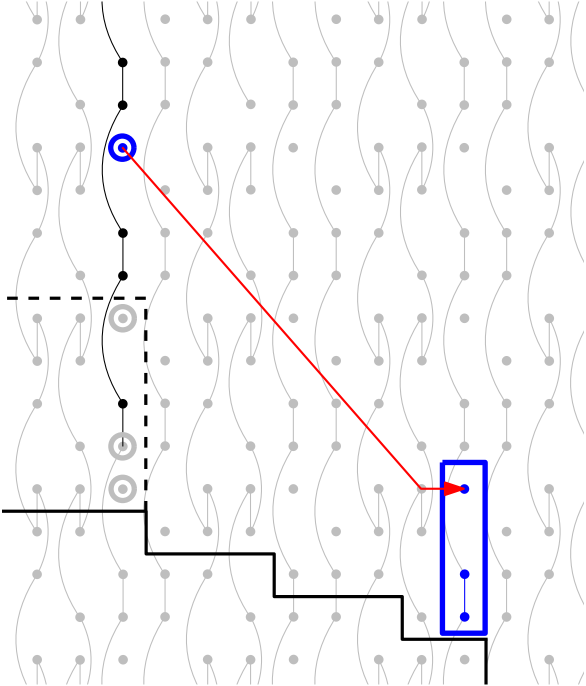

Thank you!