Topological Hochschild Cohomology for Schemes

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Work in progress!

1. Cohomology theories for rings
2. THH^* for rings
3. THH^* for enriched cat's
4. THH^* for schemes
5. Computations for $\mathbb{P}^1$ & $\mathbb{P}^2$

K a commutative base ring

A a K-algebra

$A^e = A \otimes_K A^{op}$ enveloping alg

<u>Hochschild cohomology</u> is

$$HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$$

$A^{\perp\!\!\!\perp e} = A \otimes_K^{\perp\!\!\!\perp} A^{op}$ derived env alg

<u>Shukla cohomology</u> is

$$\mathrm{Shukla}^*(A) = \mathrm{Ext}^*_{A^{\perp\!\!\!\perp e}}(A, A)$$

Deformation theory

$HH^2(A)$ is in bijection with

$$\left\{ \begin{array}{c} \text{split 'first-order' extensions of } A \end{array} \right\} \Big/ \text{iso}$$

i.e. $k[\varepsilon]/\varepsilon^2$ – algebras $\tilde{A}$

with a k-split map

$$\tilde{A} \longrightarrow \tilde{A}/\varepsilon \cong A$$

Upshot

behaves well over a field

There's a similar story for Shukla cohomology

Motivation

is there a similar story for abelian categories?

Def (Lowen)

$\mathcal{A}$ an abelian category

$$HH^*_{ab}(\mathcal{A}) = HH^*(\mathrm{Inj}\,\mathrm{Ind}\,\mathcal{A})$$

$$HH^*(X) \simeq HH^*_{ab}(\mathrm{QCoh}\,X)$$

a $\mathbb{Z}$-linear category

a.k.a a ring with many objects

$HH^{\mathbb{Z}}_{ab}$ doesn't work well when things aren't linear over a field

Example

$\text{Mod-}\mathbb{Z}/p^2$ should be a first-order deformation of $\text{Mod-}\mathbb{Z}/p$

Need to incorporate non-additive features

Answer

Mac Lane cohomology

Mac Lane cohomology

rmk: not the original def^n

Equivalence is due to Jibladze & Pirashvili

$$HML^*(A) = Ext^*_{\mathcal{F}(A)}(A, A)$$

non-additive bimodules

$$\mathcal{F}(A) = Fun(free\,A, Mod\text{-}A)$$

free f.g. A-Mods

$$A\text{-bimod} \hookrightarrow \mathcal{F}(A)$$

$$M \longmapsto M \otimes_A -$$

Eventual goal

Use HML to get a good deformation theory for abelian categories

e.g. want

$$HML^2 \simeq \left\{ \begin{matrix} 1^{st} \text{ order} \\ \text{deformations} \end{matrix} \right\}$$

Today focus on the topological approach to HML

2 Topological story

<u>Idea</u> one can & should do algebra with ring spectra

$$\Sigma X_i \longrightarrow X_{i+1}$$

Since the 90s categories of <u>highly structured</u> spectra with a monoidal smash product $\wedge$

Ring spectra are monoids in $(Sp^t, \wedge)$

Example $\mathbb{S}_n = S^n$

$\mathbb{S}$ the sphere spectrum is the unit for $\wedge$

It's a commutative ring sp & moreover the initial one.

A a ring, HA the Eilenberg-Mac Lane spectrum is a ring spectrum

$\Rightarrow \exists! \; \mathbb{S} \longrightarrow HA$

Def

$THH^*(A)$ is

$\mathrm{Ext}^*_{A \wedge_S A^{op}}(A, A)$

& similarly for THH_*

Example (Bökstedt, Franjou-Lannes-Schwartz)

k a finite field

$THH^*(k) \simeq k[u]$

deg 2

Example

If A is a $\mathbb{Q}$-algebra then $THH^*(A) \simeq HH^*(A)$

since $\mathbb{Q} \wedge_S \mathbb{Q} \simeq \mathbb{Q}$

Example

If A is an algebra over a field k then there's a spectral sequence

$$THH^q(k) \otimes HH^p(A) \Longrightarrow THH^{p+q}(A)$$

'arithmetic' 'geometric'

Thm (Pirashvili–Waldhausen 1990)

$$HML_* \cong THH_*$$

Thm

$$HML^* \cong THH^*$$

Key ingredient

Recent (yesterday!) result of Hörel & Ramzi

$$\text{Shukla}^2 = \text{HML}^2$$

$$\text{Shukla}^* \to \text{HML}^*$$

3 | THH^* for enriched categories

Def a <u>spectral category</u> is a category enriched in $(Sp, \wedge) \simeq (\mathbb{S}\text{-mod}, \wedge)$

Example

Sp itself is a spectral category (enrichment by internal homs)

Analogy

Spectral cats : ring spectrum

DG cats : DGA

One can talk about modules over a spectral category & this allows you to define $\mathcal{C} \wedge_{\mathbb{S}} \mathcal{C}^{op}$

$$THH(\mathcal{C}) = \mathbb{R}\mathrm{Hom}_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C})$$

which is a <u>spectrum</u>

<u>rmk</u> Can compute via a bar construction

$\underline{\text{Th}^{m}}$ (Tabuada)

$\exists$ a chain of Quillen equivalences

$$H: dgCat_{\mathbb{Z}} \longrightarrow H\mathbb{Z}\text{-}Cat$$

(cat) enriched over ($H\mathbb{Z}$_mod, $\wedge$)

This is a many-object version of a result of Shipley

$\mathbb{S} \xrightarrow{i} H\mathbb{Z}$ induces a restriction of scalars map

$$H\mathbb{Z}\text{-Cat} \xrightarrow{i^*} \mathbb{S}\text{-Cat}$$

<u>Def</u> $\mathcal{C}$ a dg-$\mathbb{Z}$-cat

$$THH^*(\mathcal{C}) := THH^*(i^* H\mathcal{C})$$

<u>rmk</u> If A is a ring, regarded as a 1-obj. dg category, get the expected answer

4 THH^* for schemes

<u>Def</u> $\mathcal{A}$ an abelian cat

$$THH^*_{ab}(\mathcal{A}) = THH^*(\mathrm{Inj\,Ind}\,\mathcal{A})$$

$\mathbb{Z}$-linear cat.

<u>Def</u> X a scheme

$$THH^*(X) := THH^*_{ab}(\mathrm{QCoh}\,X)$$

<u>Warning</u> Should only be expected to behave when X is qcqs

Sanity Check

If R is a commutative ring then

$$
\begin{aligned}
THH(\operatorname{Spec} R) &\simeq THH_{ab}(\operatorname{QCoh} \operatorname{Spec} R) \\
&\simeq THH_{ab}(\operatorname{Mod-}R) \\
&\simeq THH(R)
\end{aligned}
$$

true but not obvious!

Thm X qcqs noetherian

$$THH^{\not\ell}(X) \simeq THH^{\not\ell}_{ab}(\mathrm{Coh}\,X)$$

Proof

$\mathrm{QCoh}\,X = \mathrm{Ind}\,\mathrm{Coh}\,X$

so $THH^{\not\ell}(X)$ is

$THH^{\not\ell}(\mathrm{Inj}\,\mathrm{QCoh}\,X)$

$\simeq THH^{\not\ell}_{ab}(\mathrm{QCoh}\,X)$

because QCoh has enough injectives

<u>Punchline</u> X qcqs noeth.

$$
\begin{aligned}
THH(X) &\simeq THH(D^b Coh X) \\
&\simeq THH(perf X) \\
&\simeq THH(DQCoh X)
\end{aligned}
$$

<u>Proof idea</u>

Adapt the arguments for
HH^* due to via Keller
Lowen & Van den Bergh

<u>Key point</u> computing THH
via the bar complex gives
limited functoriality.

5 Computations

Def X qcqs Noetherian a <u>tilting complex</u> is a compact generator $T \in D^b \mathrm{Coh} X$ such that $\mathrm{Ext}^*(T,T)$ is concentrated in degree zero

Def Say X is tiltable if it admits a tilting complex

Example $\mathbb{P}^n$ is tiltable

$(T = \mathcal{O} \oplus \cdots \oplus \mathcal{O}(n))$

Example Grassmannians are tiltable

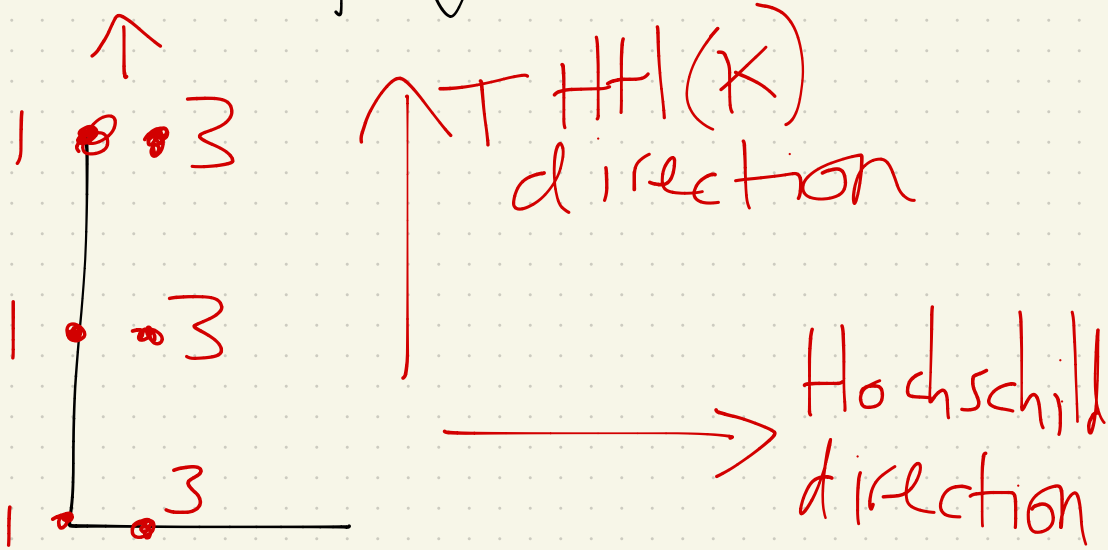
Thm X a tiltable scheme over a field k then $\exists$ a spectral seq

$$THH^q(k) \otimes HH^p(X) \Longrightarrow THH^{p+q}(X)$$

Follows from the analogue for rings

For $\mathbb{P}^1$ over a finite field K

the E^2 page is



& the SS degenerates

We get $thh^n(\mathbb{P}^1) \simeq$

$thh^n(K) \oplus HH^1(\mathbb{P}^1) \otimes thh^{n-1}(K)$

1, 3, 1, 3, 1, 3, ...

For $\mathbb{P}^2$ over a finite field we also get degeneration

Dimensions are

$1, 8, 11, 8, 11, 8, 11, \ldots$

For $\mathbb{P}^3$ we might have an E^3 differential

$$HH^0(\mathbb{P}^3) \longrightarrow HH^1(\mathbb{P}^3)$$

$\cong k$

$$THH^p(A) \otimes THML^q(K) \Longrightarrow THML^{p+q}(A)$$

Kaledin–Larsen

$$THH^*(\mathbb{Z})$$

$\mathbb{Z}/i$ deg $2i+1$

0 else