Classification of irreducible modules for Bershadsky-Polyakov algebra at certain levels

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Vertex algebras

Vertex algebra is a triple $(V, Y, \mathbb{1})$, where V is a vector space over $\mathbb{C}$, $\mathbb{1} \in V$ is the vacuum vector and Y is an operator

$$Y : V \longrightarrow (End\, V)[[z, z^{-1}]], \quad Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

satisfying following axioms:

1. $Y(a, z)b = \sum_{n \in \mathbb{Z}} a_n b z^{-n-1}$ has finitely many negative powers,
2. $Y(\mathbb{1}, z) = Id$,
3. $Y(a, z)\mathbb{1} \in V[[z]]$ and $\lim_{z \to 0} Y(a, z)\mathbb{1} = a$,
4. $[D, Y(a, z)] = \frac{d}{dz} Y(a, z))$, where $D \in End\, V$ is given by $Da = a_{-2}\mathbb{1}$,
5. $\forall\, a, b \in V$, $\exists\, N$ such that $$(z_1 - z_2)^N [Y(a, z_1), Y(b, z_2)] = 0.$$

Zhu algebra

Let $V = \bigoplus_{n=0}^{\infty} V(n)$ be a $\mathbb{Z}$-graded VOA, and let $\deg a = n$, for $a \in V(n)$.
Define bilinear mappings $* : V \times V \longrightarrow V$, $\circ : V \times V \longrightarrow V$:

$$a * b = Res_z \left(Y(a, z) \frac{(1 + z)^{\deg a}}{z} b \right),$$

$$a \circ b = Res_z \left(Y(a, z) \frac{(1 + z)^{\deg a}}{z^2} b \right),$$

for $a \in V(n)$, $b \in V$.
Let $O(V) \subset V$ be the linear span of the elements $a \circ b$. The quotient space

$$A(V) = \frac{V}{O(V)}$$

is an associative algebra called the **Zhu algebra** of the VOA V.

Let $A(\mathcal{W}^k)$ denote the Zhu algebra of $\mathcal{W}^k$. Let $[v]$ be the image of $v \in \mathcal{W}^k$ under the mapping $\mathcal{W}^k \mapsto A(\mathcal{W}^k)$.

- $A(\mathcal{W}^k)$ is generated by $[G^+], [G^-], [J], [\omega]$
- Zhu algebra $A(\mathcal{W}^k)$ is actually a quotient of another associative algebra, called Smith algebra

Classification of irreducible $\mathcal{W}_{-5/3}$-modules

- Define functions $$h_i(x, y) = \frac{1}{i}(g(x, y) + g(x + 1, y) + \ldots + g(x + i - 1, y))$$
- (Ar2013) If the top level $L(x, y)(0)$ is n-dimensional, then $h_n(x, y) = 0$.
- We will need the following Δ-operator $$\Delta(-J, z) = z^{-J(0)} exp \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{-J(k)}{k z^k} \right),$$ such that $$\sum_{n \in \mathbb{Z}} \Psi(a_n) z^{-n-1} = Y(\Delta(-J, z)a, z).$$
- (Ar2013) Let $\dim\,(L(x, y)(0)) = i$. Then $$\Psi(L(x, y)) \cong L(x + i - 1 - \frac{2k + 3}{3}, y - x - i + 1 + \frac{2k + 3}{3}).$$

Theorem

Define

$$\mathcal{R}_k = \{(-\frac{1}{9}, 0), (0, 0), (\frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, \frac{2}{3}), (-\frac{4}{9}, \frac{1}{3}), (-\frac{7}{9}, \frac{2}{3})\}.$$

$$\widetilde{\mathcal{R}_k} = \{(\frac{1}{9}, -\frac{1}{9}), (\frac{4}{9}, -\frac{1}{9}), (\frac{7}{9}, -\frac{1}{9})\}.$$

Let $k = -5/3$. The set

$$\{L(x, y) \mid (x, y) \in \mathcal{R}_k \cup \widetilde{\mathcal{R}_k}\},$$

gives a complete list of irreducible $\mathcal{W}_k$–modules from the category $\mathcal{O}$.

Sketch of proof:

- first we compute an explicit formula for the singular vector in $\mathcal{W}_{-5/3}$ at level 4
- from this formula, we obtain a relation in the Zhu algebra $A(\mathcal{W}_k)$: $$[G^+]^2([\omega] + \frac{1}{9}) = 0$$
- using this relation (and applying the above Δ-operator), we get candidates for highest weight $\mathcal{W}_k$–modules
- in order to obtain a realization of those modules, we show that $\mathcal{W}_k$ can be realized as a $\mathbb{Z}_3$-orbifold (fixed points subalgebra) of the Weyl vertex algebra W.

Bershadsky-Polyakov vertex algebra

Minimal affine $\mathcal{W}$–algebra $\mathcal{W}^k(\mathfrak{g}, f_\theta)$, where f_θ is a minimal nilpotent element, is the vertex algebra obtained by quantum Drinfeld-Sokolov reduction from the affine vertex algebra $V^k(\mathfrak{g})$.

Vertex algebra $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ is strongly generated by vectors

- $G^{\{u\}}$, $u \in \mathfrak{g}_{-\frac{1}{2}}$, of conformal weight $\frac{3}{2}$
- $J^{\{a\}}$, $a \in \mathfrak{g}^{\natural}$, of conformal weight 1
- ω is the conformal vector of central charge $$c(\mathfrak{g}, k) = \frac{k \dim \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

For $k \neq -h^\vee$, $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ has a unique simple quotient $\mathcal{W}_k(\mathfrak{g}, f_\theta)$.

Bershadsky-Polyakov vertex algebra $\mathcal{W}^k := \mathcal{W}^k(sl_3, f_\theta)$ is the minimal affine $\mathcal{W}$-algebra obtained by quantum Drinfeld-Sokolov reduction from $V^k(sl_3)$.

- $\mathcal{W}^k$ is generated by the fields T, J, G^+, G^-
- we choose a new Virasoro vector $$L(z) = T(z) + \frac{1}{2} DJ(z)$$
- the fields L, J, G^+, G^- satisfy commutation relations:

$[J(m), J(n)] = \frac{2k+3}{3} m \delta_{m+n,0}$, $\quad [J(m), G^{\pm}(n)] = \pm G^{\pm}(m + n)$,

$[L(m), J(n)] = -nJ(m + n) - \frac{(2k+3)(m+1)m}{6} \delta_{m+n,0}$,

$[L(m), G^+(n)] = -nG^+(m + n)$, $\quad [L(m), G^-(n)] = (m - n)G^-(m + n)$,

$[G^+(m), G^-(n)] = 3(J^2)(m + n) + (3(k + 1)m - (2k + 3)(m + n + 1))J(m + n) - (k + 3)L(m + n) + \frac{(k+1)(2k+3)(m-1)m}{2} \delta_{m+n,0}$.

Smith-type algebra

Let $g(x, y) \in \mathbb{C}[x, y]$ be an arbitrary polynomial. Associative algebra $R(g)$ of **Smith type** is generated by $\{E, F, X, Y\}$ such that Y is a central element and the following relations hold:

$$XE - EX = E, \; XF - FX = -F, \; EF - FE = g(X, Y).$$

- $R(g)$ is a certain generalization of $U(sl_2)$!

Structure of the Zhu algebra $A(\mathcal{W}^k)$

Denote $E = [G^+]$, $F = [G^-]$, $X = [J]$, $Y = [\omega]$. Let $R(g)$ be the Smith-type algebra generated by $\{E, F, X, Y\}$, with

$$g(x, y) = -(3x^2 - (2k + 3)x - (k + 3)y).$$

Then the Zhu algebra $A(\mathcal{W}^k)$ associated to the Bershadsky-Polyakov algebra $\mathcal{W}^k$ is isomorphic to a certain quotient of the Smith algebra $R(g)$.

Irreducible $\mathcal{W}_k$-modules for integer levels k

Let $L(x, y)$ be the irreducible highest weight $\mathcal{W}_k$-module of weight $(x, y) \in \mathbb{C}^2$.

- vectors $$(G^+(-1))^n \mathbb{1}, \; (G^-(-2))^n \mathbb{1}$$ are singular in $\mathcal{W}^k$ for $n = k + 2$, where $k \in \mathbb{Z}$.

Necessary condition for $\mathcal{W}_k$-modules

Let $k \in \mathbb{Z}$, $k \geq -1$, $(x, y) \in \mathbb{C}^2$. Then we have:

(i) The set of equivalency classes of irreducible ordinary $\mathcal{W}_k$–modules is contained in the set

$$\mathcal{S}_k = \{L(x, y) \mid h_i(x, y) = 0, \; 1 \leq i \leq k + 2\}.$$

(ii) Every irreducible $\mathcal{W}_k$–module in the category $\mathcal{O}$ is an ordinary module.

- question: are modules from the set $\mathcal{S}_k$ indeed $\mathcal{W}_k$-modules?

Conjecture

The set $\{L(x, y) \mid (x, y) \in \mathcal{S}_k\}$ is the set of all irreducible ordinary $\mathcal{W}_k$–modules.

We prove this conjecture for $k = -1$ and $k = 0$, and classify all modules in the category $\mathcal{O}$.

References

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