Naruse Hook length formula for linear extensions of mobile posets

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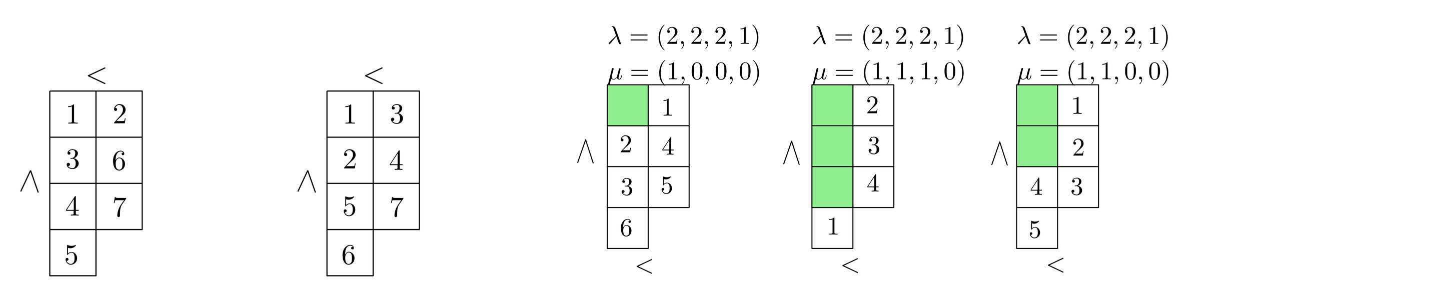
Objective

- Extend the Naruse Hook Length Formula to *mobile posets*
- Find a q-analogue of the NHLF for mobile posets

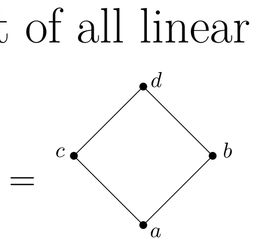
Standard Young Tableaux

- A standard Young tableau (SYT) is a filling of λ with $1,\dots,n$ such that it is increasing in rows and columns.
- A skew shape is a pair of partitions (λ,μ) such that $\mu\subseteq\lambda$, denoted as λ/μ. A *skew SYT* is a filling of λ/μ with integers $1\dots n$ increasing in rows and columns.



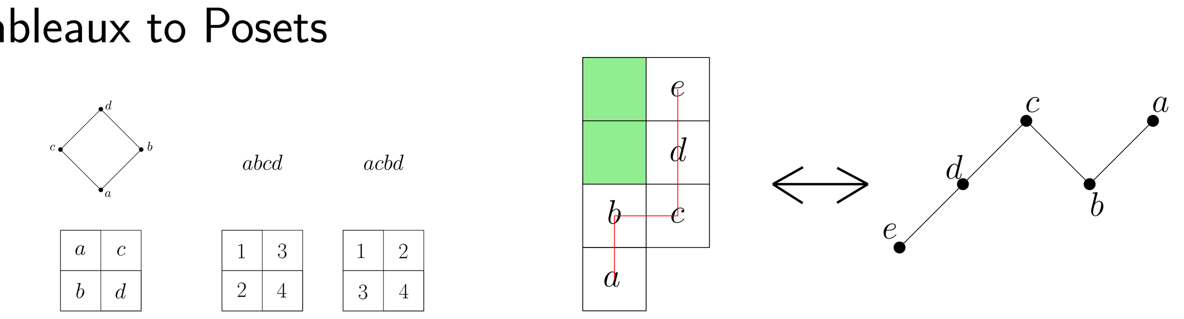
Linear Extensions

A linear extension of a poset P is a linear order of elements compatible with the order P. Let $\mathcal{L}(P)$ the set of all linear extensions of P and $e(P)=|\mathcal{L}(P)|$.

$P=$  $\mathcal{L}(P)=\{\ abcd,\ acbd\}$

$e(P)=2$

From to Tableaux to Posets



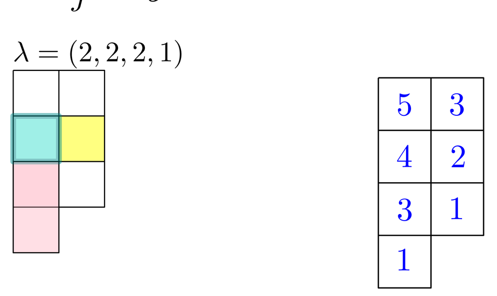
$$e(P(\lambda/\mu))=|\mathrm{SYT}(\lambda/\mu)|$$

Hook Length Formula

Theorem 1. (Frame-Robinson-Thrall, 1954) Let λ be a partition of n. We have

$$|\mathrm{SYT}(\lambda)|=n!\prod_{(i,j)\in[\lambda]}\frac{1}{h(i,j)}$$

where $h(i,j)=\lambda_i-i+\lambda'_j-j+1$ is the hook length of the square (i,j)



$7!\cdot\frac{1}{5\cdot4\cdot3\cdot3\cdot2\cdot1\cdot1}=14$

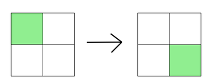
Naruse Hook Length Formula

Theorem 2. (Naruse, 2014) For a skew shape λ/μ, we have

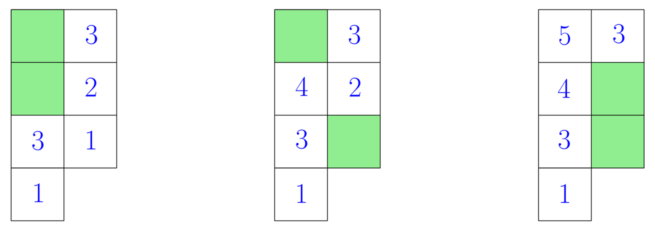
$$|\mathrm{SYT}(\lambda/\mu)| = |\lambda/\mu|!\sum_{D\in\mathcal{E}(\lambda/\mu)}\prod_{u\in[\lambda]\setminus D}\frac{1}{h(u)}.$$

Excited Diagram (Ikeda-Naruse)

An excited move in λ/μ is a sliding move of active cell from (i,j) to $(i+1,j+1)$.



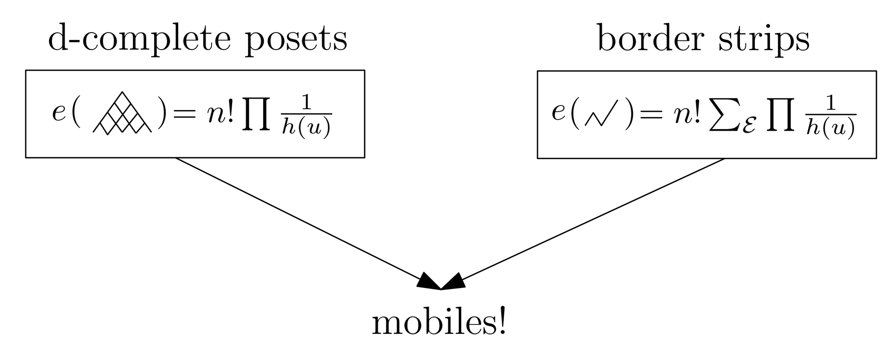
An excited diagram of λ/μ is a subdiagram of $[\lambda]$ obtained from the Young diagram of μ after a sequence of excited moves.



$$|\mathrm{SYT}((2,2,2,1),(1,1))| = 5!\left(\frac{1}{3\cdot3\cdot2\cdot1\cdot1}+\frac{1}{4\cdot3\cdot3\cdot2\cdot1}+\frac{1}{5\cdot4\cdot3\cdot3\cdot1}\right) = 9$$

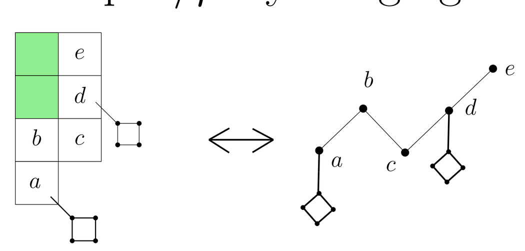
Mobile Posets

Question: Are there other posets with Naruse hook length formula, generalizing formula for $e(P)$?



. (Proctor) *d-complete posets* are a large class of posets that includes (shifted) Young diagrams and rooted tree posets with HLF

. (Garver-Grosser-Matherne-Morales, 2020) A (free-standing) mobile poset P is a poset obtained from a border strip λ/μ by hanging from it $d-$complete posets.



Determinant Formula

Theorem 3. (G-G-M-M, 2020) Let P be a mobile tree poset with n elements then

$$e(P)=n!\det(M_{i,j})$$

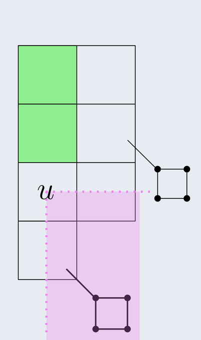
where $M_{i,j}=0$ if $j<i-1$, $M_{i,j}=1$ if $=<i-1$, $M_{i,j}=1/\prod_{x\in P_{i,j}}h_{P i,j}(x)$

NHLF for Mobiles

Theorem 4. (P, 2020+) Let $P_{\lambda/\mu}(\mathbf{p})$ be a free-standing mobile poset of size n then

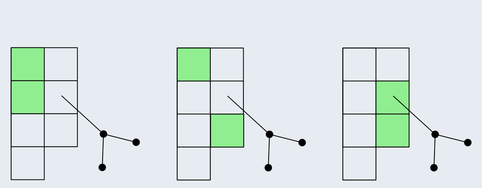
$$e(P_{\lambda/\mu}(\mathbf{p}))=\frac{n!}{\prod_{u\in\mathbf{p}}h(u)}\sum_{D\in\mathcal{E}(\lambda/\mu)}\prod_{(i,j)\in\overline{D}}\frac{1}{h'(i,j)}$$

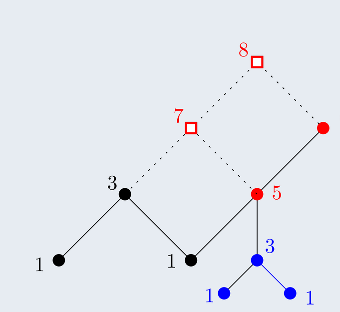
where $p_{a,b}$ is the size of a d-complete poset hanging on (a,b)



$$h'(i,j)=h(i,j)+\sum_{a\geq i,b\geq j}p_{a,b}$$

$$h'(u)=3+4$$





$$e(P)=\frac{8!}{1\cdot1\cdot3}\left(\frac{1}{1\cdot1\cdot3\cdot5\cdot6}+\frac{1}{1\cdot3\cdot5\cdot6\cdot7}+\frac{1}{1\cdot3\cdot6\cdot7\cdot8}\right) = 184$$

Method of Proof

The case of Border Strips (Morales-Pak-Panova, 2019)

$$F_{\lambda/\mu}(\mathbf{x}|\mathbf{y}):=\sum_{D\in\mathcal{E}(\lambda/\mu)}\prod_{(i,j)\in[\lambda]\setminus D}\frac{1}{x_i-y_j}$$

$$\text{Let } x_i=\lambda_i-i+1-\sum_{a<i}p_{a,b},\text{ and } y_j=j-\lambda'_j-\sum_{b\geq j}p_{a,b}$$

$$e(P(\lambda/\mu))=\sum_{\mu\to\nu}e(P(\lambda/\nu))$$

where Let ν is the shape obtained by adding an inner corner of λ/μ.

Lemma 1. Pieri-Chevalley formula for border strips

$$F_{\lambda/\mu}(\mathbf{x}|\mathbf{y})=\frac{1}{x_1-y_1}\sum_{\mu\to\nu}F_{\lambda/\nu}(\mathbf{x}|\mathbf{y})$$

Skewed d-complete Poset (RPP)

Theorem 5. (Naruse-Okada, 2019) Let P be a d-complete poset and F an order filter of P. Then the multivariate generating function of $(P\setminus F)$-partition is

$$\sum_{D\in\mathcal{E}(F)}\frac{\prod_{v\in B(D)}\mathbf{z}[H_P(v)]}{\prod_{v\in P\setminus D}(1-z[H_{(v)}])}$$

where $B(D)$ is a set of excited peaks.

A SSYT q-analogue

Theorem 6. (Morales, Pak, Panova 15)

$$s_{\lambda/\mu}(1,q,q^2,\dots)=\sum_{S\in\mathcal{E}(\lambda/\mu)}q^{a(D)}\prod_{(i,j)\in[\lambda]\setminus S}\frac{1}{1-q^{h(i,j)}}$$

where $a(D)=\sum_{u\in Br(D)}h(u)$ is the sum of hook-lengths of the supports of broken diagonals.

Current/Future Work

For a labeled free-standing mobile $(P_{\lambda/\mu},\omega)$, we have

$$\frac{e_q(P_{\lambda/\mu},\omega)}{\prod_{i=1}^{n}(1-q^i)}=\prod_{v\in\mathbf{p}}\frac{1}{1-q^{h(v)}}\sum_{D\in\mathcal{E}(\lambda/\mu)}q^{a'(D)}\prod_{u\in\lambda D}\frac{1}{1-q^{h'(u)}}$$

and $a'(D)=\sum_{u\in Br(D)}h'(u)$ is the sum of hook-lengths of the supports of broken diagonals.

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- Alejandro H Morales, Igor Pak, and Greta Panova, *Hook formulas for skew shapes ii. combinatorial proofs and enumerative applications*, SIAM Journal on Discrete Mathematics **31** (2017), no. 3, 1953–1989.
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- Hiroshi Naruse and Soichi Okada, *Skew hook formula for d-complete posets via equivariant K-theory*, Algebr. Comb. **2** (2019), no. 4, 541–571. MR 3997510
- GaYee Park, *Naruse hook formula for linear extensions of mobile posets (in preparation).*