Promotion, Webs, and Kwebs

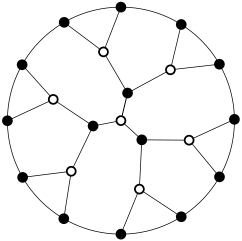
Rebecca Patrias

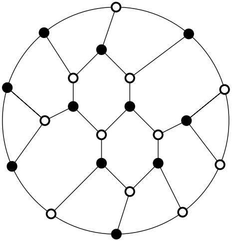
Dynamical Algebraic Combinatorics

October 21, 2020

This talk is being recorded.

Webs





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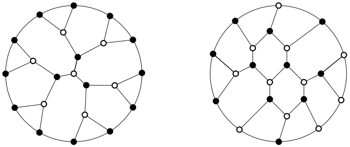

Definition (Kuperberg)

An **irreducible web** is a planar, directed graph D with no multiple edges embedded in a disk satisfying the following conditions:

1. D is bipartite,
2. all of the boundary vertices have degree 1,
3. all internal vertices have degree 3, and
4. all internal faces of D have at least 6 sides.

Web Invariants

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Web invariants

Web invariant $[D]$ is invariant under an $SL(3)$ action.

Consider the matrices $y = (y_{ij})$ and $x = (x_{ij})$. For any $g \in SL(3)$, $[D]$ is invariant under the transformation that simultaneously replaces

- x with gx and
- y with yg^{-1}.

Theorem (Kuperberg)

Let V be a 3-dimensional complex vector space. Web invariants with a fixed boundary pattern with a white vertices and b black vertices form a basis in the ring of invariants $\mathbb{C}[(V^*)^a \times V^b]^{SL(V)}$.

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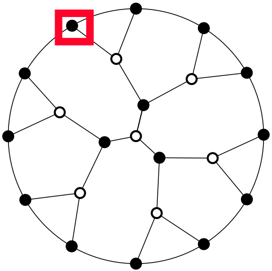
S. Fomin and P. Pylyavskyy constructed a cluster algebra structure on the ring of invariants that interacts well with the web basis in most cases.

Webs and SYT

There is a bijection between webs with n cyclically labeled, black boundary vertices and 3-row, rectangular standard Young tableaux with n boxes. (Khovanov–Kuperberg)

- Make a proper edge coloring using the following preference:
  - $\bullet$ prefers 1, then 0, then -1
- Look at edge colors adjacent to boundary vertices.
  - 1 means top row
  - 0 means middle row
  - -1 means bottom row

  (Bazier-Matte–Douville–Garver–P.–Thomas–Yildirim)



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- From left to right, connect entry y with the largest entry in the row above that is $\leq y$.
- Form corresponding tripods.
- Resolve crossings.
  (Tymoczko)

1	2	4
3	5	7
6	8	9

Theorem (Petersen–Pylyavskyy–Rhoades)

Let D be a web with cyclically labeled boundary vertices and all black boundary vertices. The standard Young tableau associated with counterclockwise rotation of D is given by promotion of the tableau associated with D itself.

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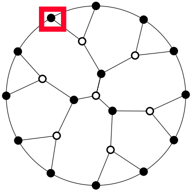

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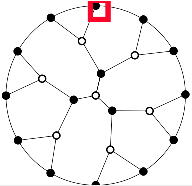

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Corollary: Let $p(T)$ denote the promotion of a rectangular, 3-row standard Young tableau with n boxes. Then $p^n(T) = T$.

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Theorem (Russell, P.)

Let D be a web with cyclically labeled boundary vertices. Web rotation corresponds to semistandard/generalized oscillating tableau promotion.

K-Promotion on increasing tableaux

Recall from Oliver's talk:

An *increasing tableau* has strictly increasing rows and columns.

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Q: What are the orbit sizes of rectangular increasing tableaux under K-promotion?

Theorem (P.–Pechenik)

Let T be an $a \times b$ rectangular increasing tableau with largest entry q, and suppose the K-promotion orbit of T has cardinality k. Then k shares a prime divisor with q. (Unless $q = a + b - 1$, in which case $k = 1$.)

Conjecture: If T is a 3-row, rectangular, increasing tableau with largest entry q, the K-promotion orbit of T has cardinality dividing q.

Wouldn't it be nice if we could make some webs corresponding to increasing tableaux so that K-promotion corresponds to web rotation?

On-going work with Oliver Pechenik, Jessica Striker, and Julianna Tymoczko

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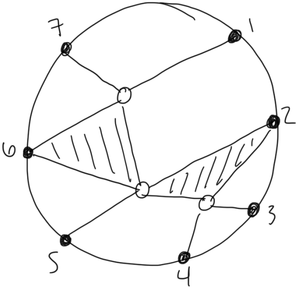

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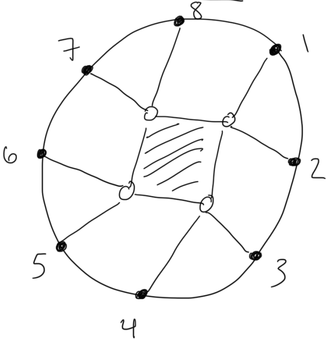

Thank you!