A birational lifting of the Lalanne–Kreweras involution on Dyck paths

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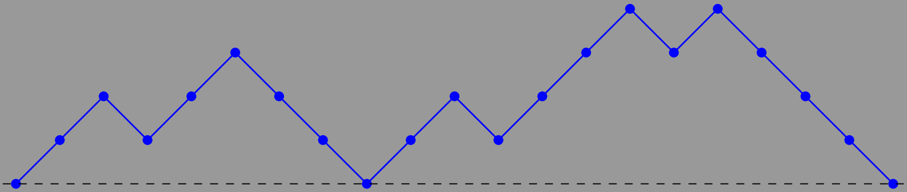
Definition

A *Dyck path of semilength n* is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ consisting of up steps $(1,1)$ and down steps $(1,-1)$ that never goes below the x-axis.

Example (Dyck path of semilength 10)

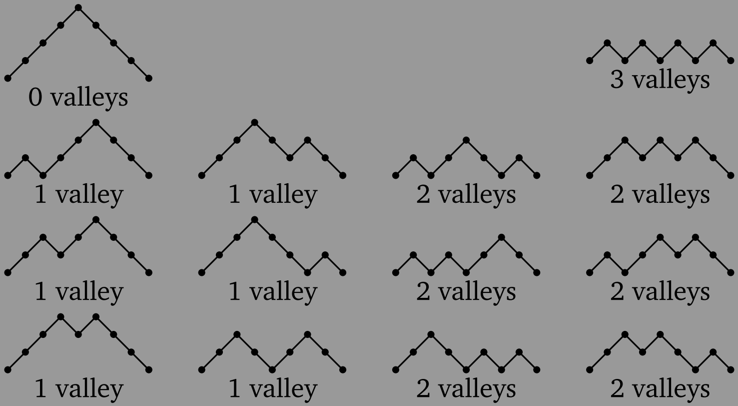


The number of Dyck paths of semilength n is $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

Valleys

Definition

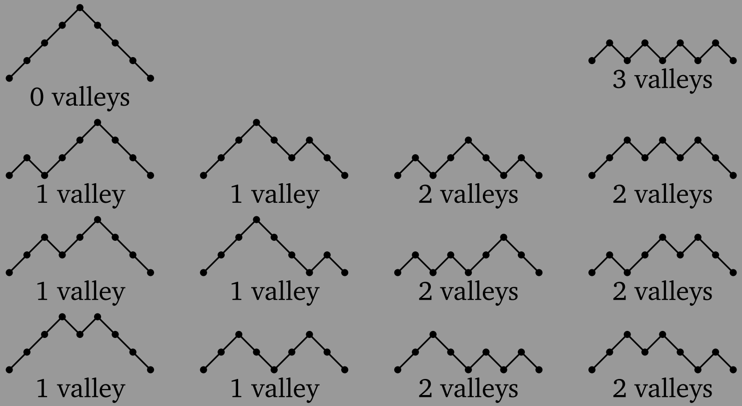
A *valley* is a $\searrow$ step immediately followed by a $\nearrow$ step.



Valleys

Definition

A *valley* is a ↘ step immediately followed by a ↗ step.

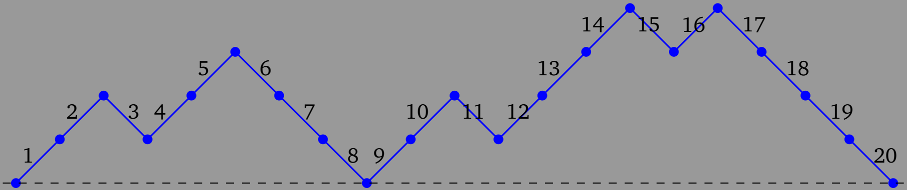


valleys	0	1	2	3
# Dyck paths	1	6	6	1

Major index

Consider the positions (from left to right) of each ↘ followed by a ↗ step. The sum of these is the *major index* of the Dyck path.

Example ($n = 10$)



 # valleys: 4 **major index:** $3 + 8 + 11 + 15 = 37$

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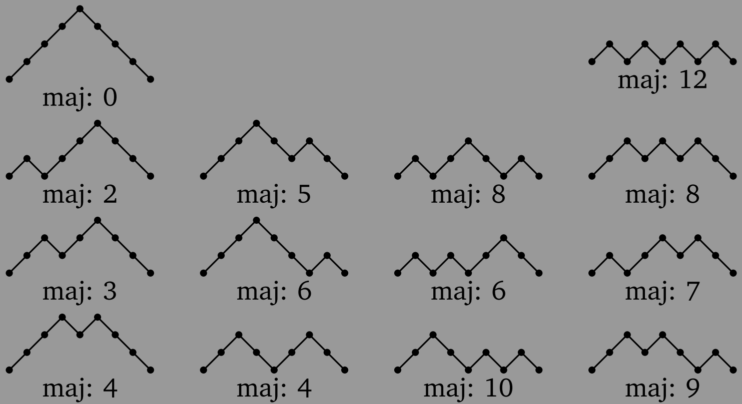

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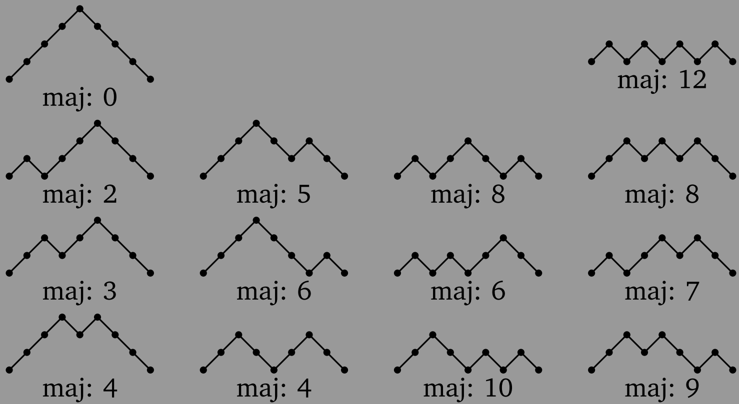


major index	0	1	2	3	4	5	6	7	8	9	10	11	12
# Dyck paths	1	0	1	1	2	1	2	1	2	1	1	0	1

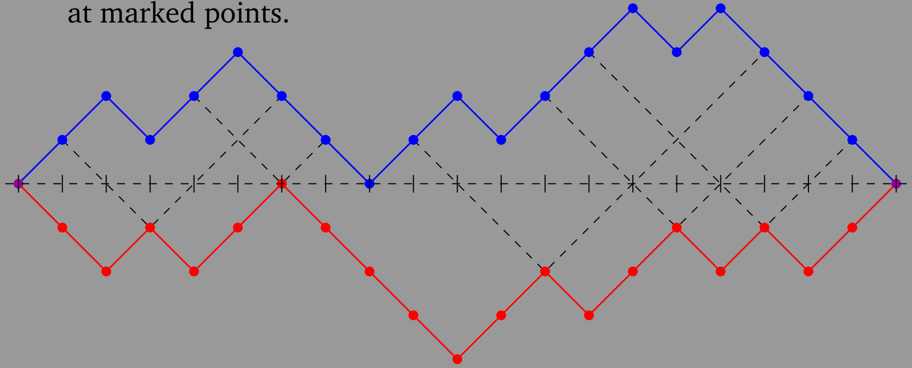
The Lalanne–Kreweras involution on Dyck paths

Lalanne and Kreweras studied an involution $\mathrm{LK} : \mathrm{Dyck}_n \to \mathrm{Dyck}_n$ for which:

1. If $p \in \mathrm{Dyck}_n$ has v valleys, then $\mathrm{LK}(p)$ has $n - 1 - v$ valleys.
2. If $p \in \mathrm{Dyck}_n$ has major index m, then $\mathrm{LK}(p)$ has major index $n(n-1) - m$.

The Lalanne–Kreweras involution on Dyck paths

1. Take a Dyck path p.
2. Draw southeast line from each junction of consecutive $\nearrow$ steps.
3. Draw southwest line from each junction of consecutive $\searrow$ steps.
4. Mark the intersection between kth (from left-to-right) southwest line and the kth southeast line.
5. $\mathrm{LK}(p)$ is the unique Dyck path (drawn upside-down) with valleys at marked points.



Symmetry of valley and major index statistics

1. If $p \in \mathrm{Dyck}_n$ has v valleys, then $\mathrm{LK}(p)$ has $n-1-v$ valleys.
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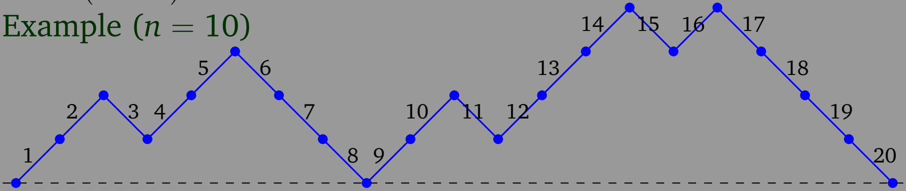
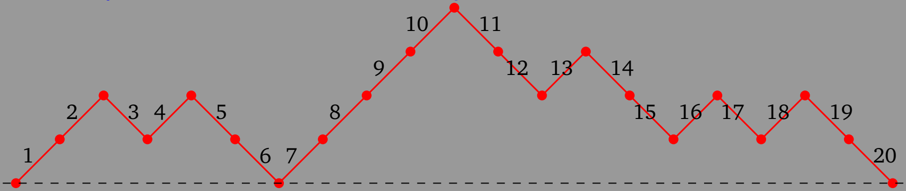


 # valleys: 4 **major index:** $3+8+11+15=37$



 # valleys: 5 **major index:** $3+6+12+15+17=53$

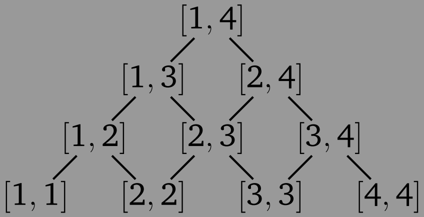
Antichains of the root poset A_n

It will be easier to study the Lalanne–Kreweras involution on the set of antichains of the type A root poset.

Definition

The elements of the *type A root poset* A_n are the intervals $[i,j] \subseteq [n] := \{1, 2, \ldots, n\}$, ordered by containment.

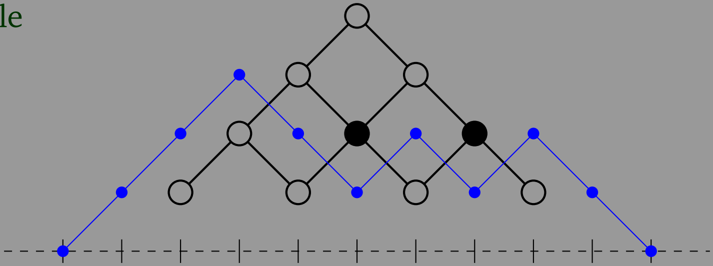
Example (A_4)



Antichains of the root poset A_n

There is a simple bijection between Dyck paths of semilength $n+1$ and *antichains* of A_n.

Example



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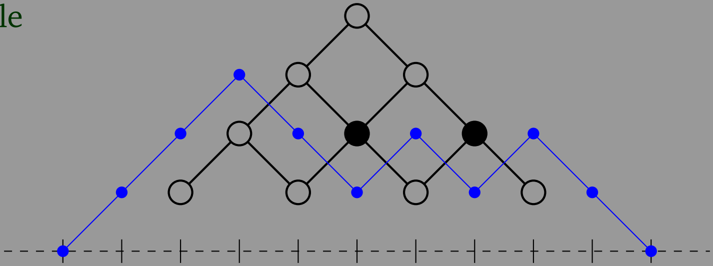

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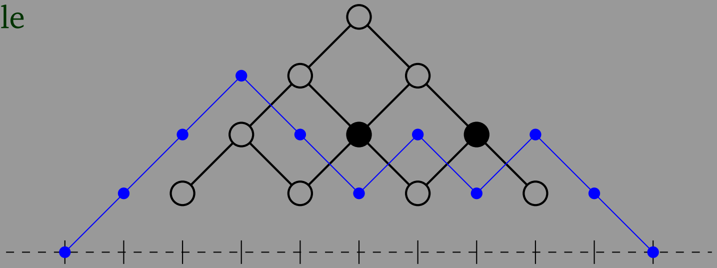
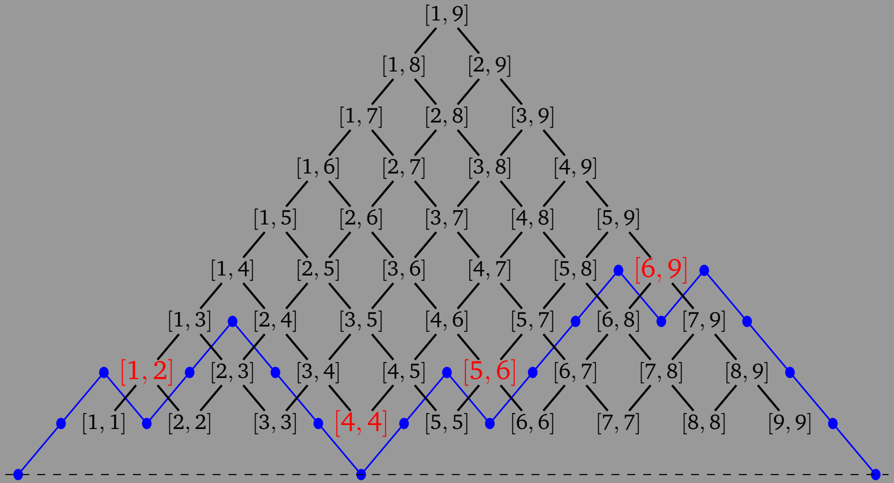


1. # valleys of Dyck path = cardinality of antichain.
2. The major index of the antichain A is $\mathrm{maj}(A) = \sum_{[i,j]\in A}(i+j)$.

Lalanne–Kreweras involution described on antichains of A_n



$$\mathrm{maj}(A) = \sum_{[i,j]\in A} (i+j) = (1+2) + (4+4) + (5+6) + (6+9) = 37$$

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Proposition

If $A = \{[i_1, j_1], \ldots, [i_k, j_k]\}$ *is an antichain of* A_n*, then* $\mathrm{LK}(A) = \{[i'_1, j'_1], \ldots, [i'_{n-k}, j'_{n-k}]\}$ *where*

$$\{i'_1, \ldots, i'_{n-k}\} := [n] \setminus \{j_1, \ldots, j_k\},$$
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Example ($n = 9$)

- $A = \{[1,2],[4,4],[5,6],[6,9]\}$

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- $A = \{[1, 2], [4, 4], [5, 6], [6, 9]\}$
- $\{i'_1, i'_2, i'_3, i'_4, i'_5\} = \{1, 3, 5, 7, 8\}$

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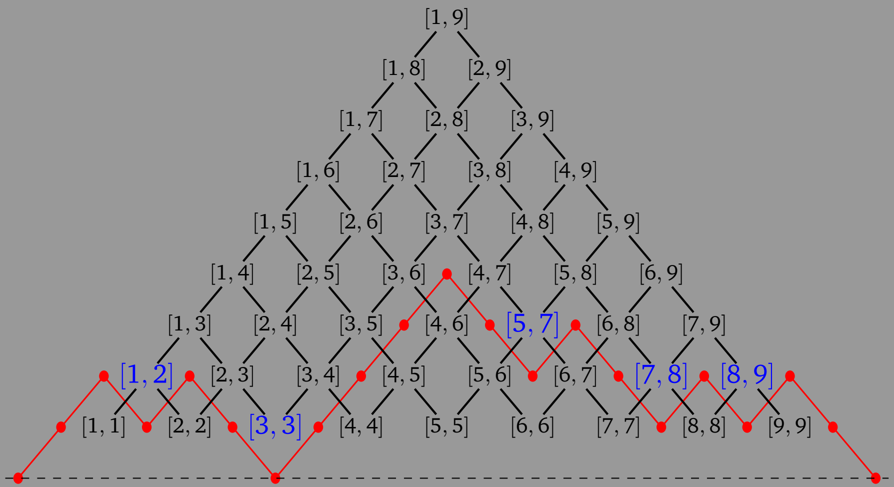
- $\{j'_1, j'_2, j'_3, j'_4, j'_5\} = \{2,3,7,8,9\}$
- $\mathrm{LK}(A) = \{[1,2],[3,3],[5,7],[7,8],[8,9]\}$

Panyushev called $\mathrm{LK}(A)$ the *dual antichain* of A, apparently unaware this same involution was studied by Lalanne and Kreweras on Dyck paths.

Lalanne–Kreweras involution described on antichains of A_n



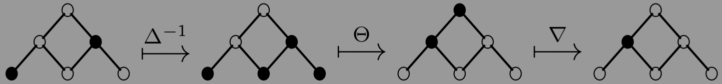
$$\mathrm{maj}(\mathrm{LK}(A)) = \sum_{[i,j]\in \mathrm{LK}(A)} (i+j) = (1+2)+(3+3)+(5+7)+(7+8)+(8+9) = 53$$

Rowmotion on antichains

Antichain rowmotion $\mathrm{Row}_{\mathcal{A}} : \mathcal{A}(P) \to \mathcal{A}(P)$ is a map on the set $\mathcal{A}(P)$ of antichains of a poset P.

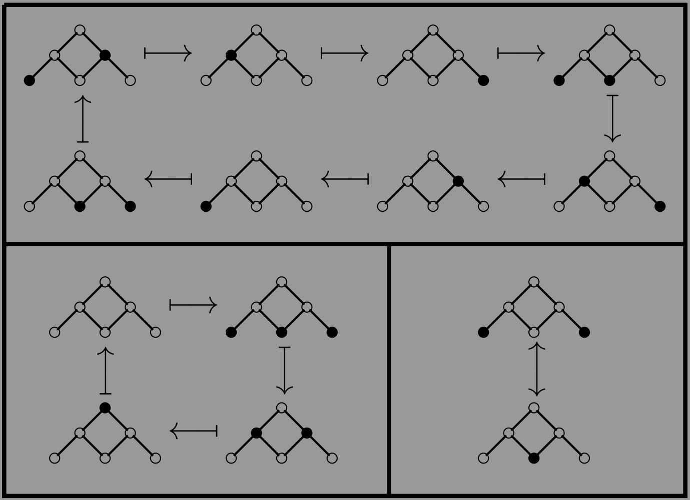
1. Δ^{-1}: Saturate downward (giving an order ideal)
2. Θ: Take the complement (giving an order filter)
3. ∇: Take the minimal elements (giving an antichain)

$\mathrm{Row}_{\mathcal{A}}$:



Rowmotion on antichains

1. On A_n, $\mathrm{Row}_{\mathcal{A}}^{2(n+1)}$ is the identity.
2. On A_n, $\mathrm{Row}_{\mathcal{A}}^{n+1}$ is reflection across the center vertical line.



Toggles

Definition

Let $e \in P$. Then the *antichain toggle* corresponding to e is the map $\tau_e : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_e(A) = \begin{cases} A \cup \{e\} & \text{if } e \notin A \text{ and } A \cup \{e\} \in \mathcal{A}(P), \\ A \setminus \{e\} & \text{if } e \in A, \\ A & \text{otherwise.} \end{cases}$$

Proposition

- *Let P be a graded poset of rank r.*
- $P_i = \{v \in P : \mathrm{rk}(v) = i\}$.
- $\boldsymbol{\tau}_i = \prod_{v \in P_i} \tau_v$
- $\mathrm{Row}_{\mathcal{A}} = \boldsymbol{\tau}_r \circ \cdots \circ \boldsymbol{\tau}_1 \circ \boldsymbol{\tau}_0$

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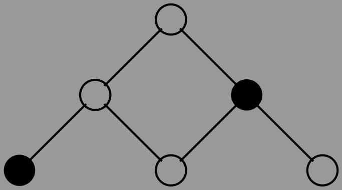

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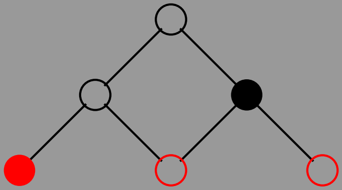

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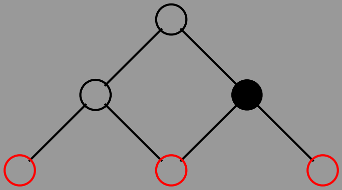

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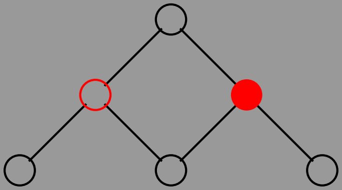

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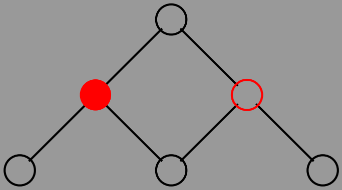

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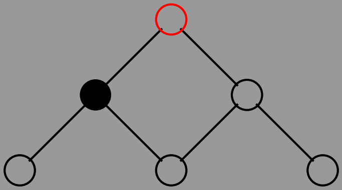

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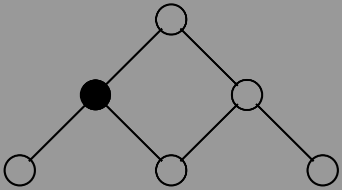

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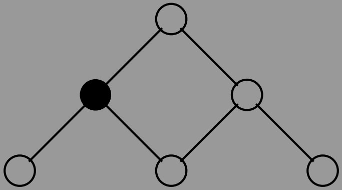
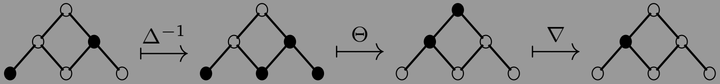


$\mathrm{Row}_{\mathcal{A}}$:



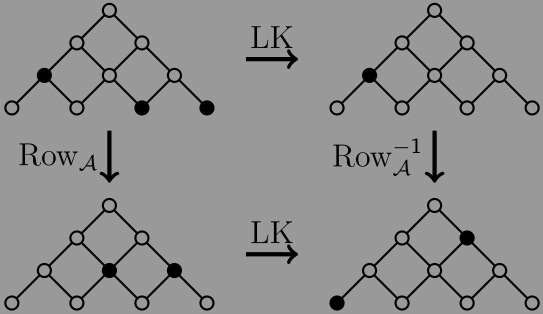
Dihedral action

Proposition (Panyushev 2009)

$$\mathrm{LK} \circ \mathrm{Row}_{\mathcal{A}} = \mathrm{Row}_{\mathcal{A}}^{-1} \circ \mathrm{LK}$$

The cyclic group action of rowmotion extends to a dihedral group action generated by $\mathrm{Row}_{\mathcal{A}}$ and LK.

Example



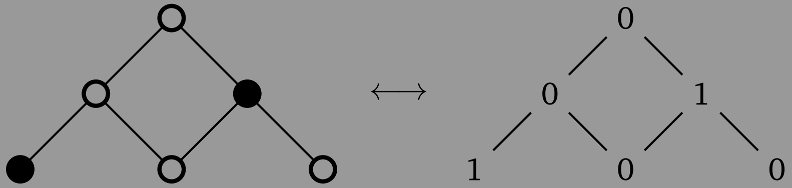
Goal

Our goal was to find a lifting of the Lalanne–Kreweras involution on A_n to the piecewise-linear and birational realms, such that the four main properties generalize to these realms.

1. LK^2 is the identity.
2. $\mathrm{LK} \circ \mathrm{Row}_{\mathcal{A}} = \mathrm{Row}_{\mathcal{A}}^{-1} \circ \mathrm{LK}$
3. $\mathrm{card}(A) + \mathrm{card}(\mathrm{LK}(A)) = n$
4. $\mathrm{maj}(A) + \mathrm{maj}(\mathrm{LK}(A)) = n(n+1)$

Chain polytope

We can associate an *indicator function* to any subset of P.



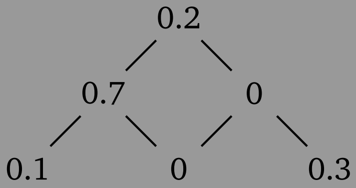
The convex hull of $\mathcal{A}(P)$ is Stanley's *chain polytope* $\mathcal{C}(P)$.

Chain polytope

Definition (Stanley 1986)

The *chain polytope* of P is the set $\mathcal{C}(P)$ of $f \in [0,1]^P$ such that $\sum_{i=1}^{n} f(x_i) \leq 1$ for all chains $x_1 < x_2 < \cdots < x_n$.

Example



$\in \mathcal{C}(\mathsf{A}_3)$

Piecewise-linear antichain toggle

Definition (J. 2017)

For $g \in \mathcal{C}(P)$, $e \in P$, $\tau_e(g)$ can only differ from g at the value of e.

$$\big(\tau_e(g)\big)(e) = 1 - \max\left\{ \sum_{i=1}^{k} g(y_i) \,\middle|\, \begin{array}{c} (y_1,\ldots,y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}$$

Piecewise-linear antichain rowmotion (or *chain polytope rowmotion*) is given by

- $P_i = \{v \in P : \mathrm{rk}(v) = i\}$.
- $\boldsymbol{\tau}_i = \prod_{v \in P_i} \tau_v$
- $\mathrm{Row}_{\mathcal{C}} = \boldsymbol{\tau}_r \circ \cdots \circ \boldsymbol{\tau}_1 \circ \boldsymbol{\tau}_0$

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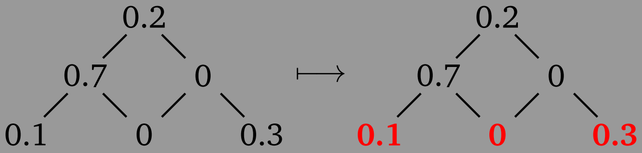

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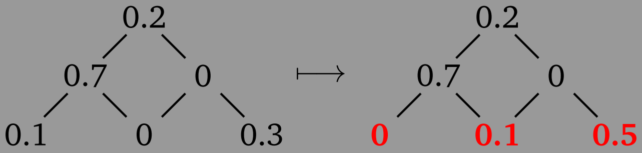

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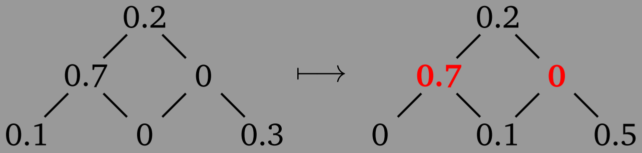
$$\big(\tau_e(g)\big)(e) = 1 - \max\left\{ \sum_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}$$

Example



Piecewise-linear antichain toggle

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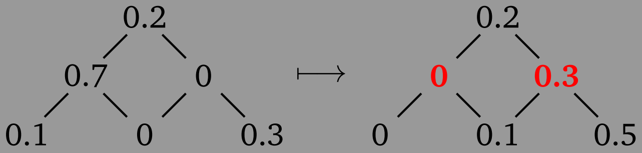

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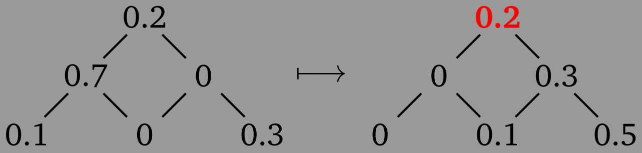

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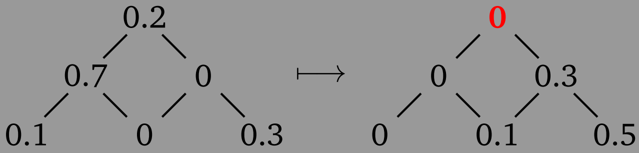

Piecewise-linear and birational dynamics

Piecewise-linear and birational toggling and rowmotion were originally defined by Einstein and Propp in 2013.

Detropicalization: from the piecewise-linear to the birational realm

	max	$+$	$-$	0	1
Replace with	$+$	$\cdot$	$/$	1	C

Definition

For $e \in P$, the *birational antichain toggle* τ_e is:

$$\left(\tau_e(g)\right)(x) = \begin{cases} \dfrac{C}{\displaystyle\sum_{(y_1,\ldots,y_k)\in \mathrm{MC}_e(P)} g(y_1)\cdots g(y_k)} & \text{if } x = e \\ g(x) & \text{if } x \neq e \end{cases}$$

Birational antichain rowmotion (BAR-motion) is given by

- $P_i = \{v \in P : \mathrm{rk}(v) = i\}$.
- $\boldsymbol{\tau}_i = \prod_{v \in P_i} \tau_v$
- $\mathrm{BAR} = \boldsymbol{\tau}_r \circ \cdots \circ \boldsymbol{\tau}_1 \circ \boldsymbol{\tau}_0$

Birational antichain rowmotion

$$\left(\tau_e(g)\right)(x) = \begin{cases} \dfrac{C}{\displaystyle\sum_{(y_1,\ldots,y_k)\in \mathrm{MC}_e(P)} g(y_1)\cdots g(y_k)} & \text{if } x = e \\ g(x) & \text{if } x \neq e \end{cases}$$

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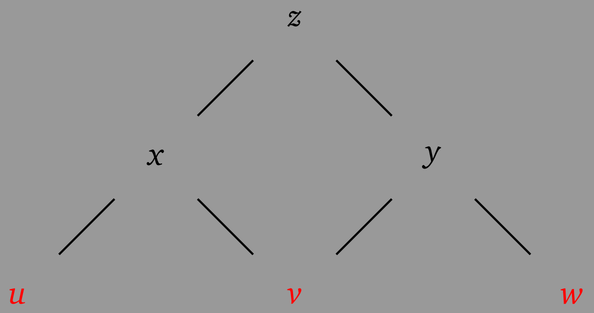

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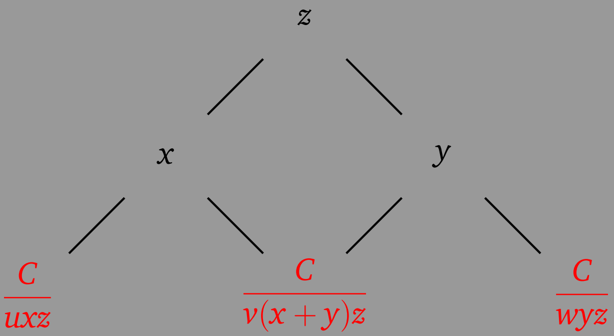

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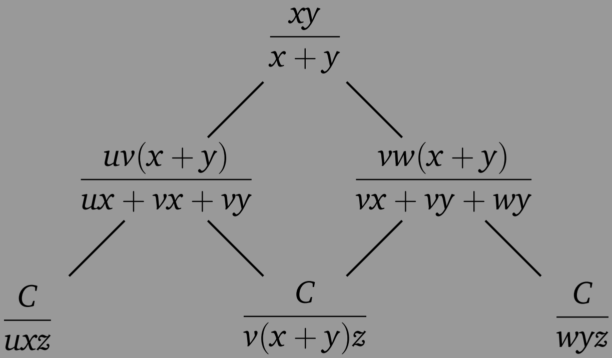
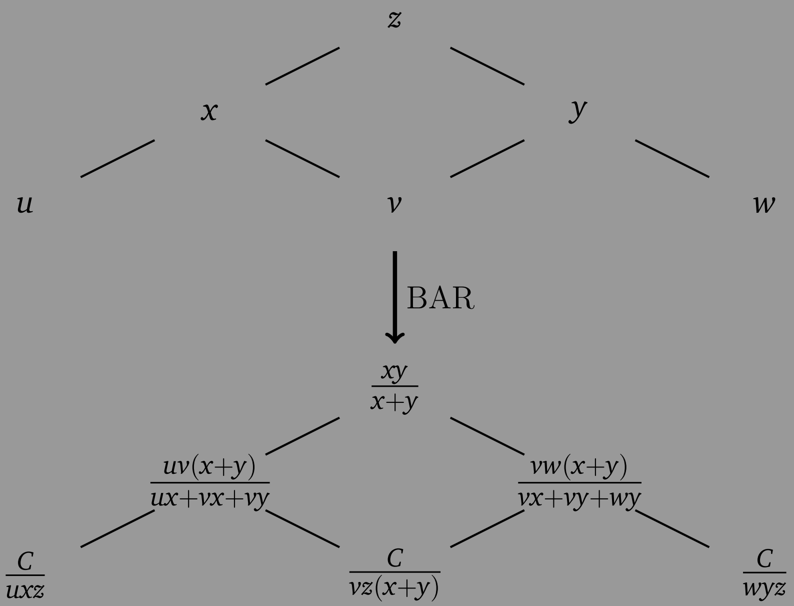
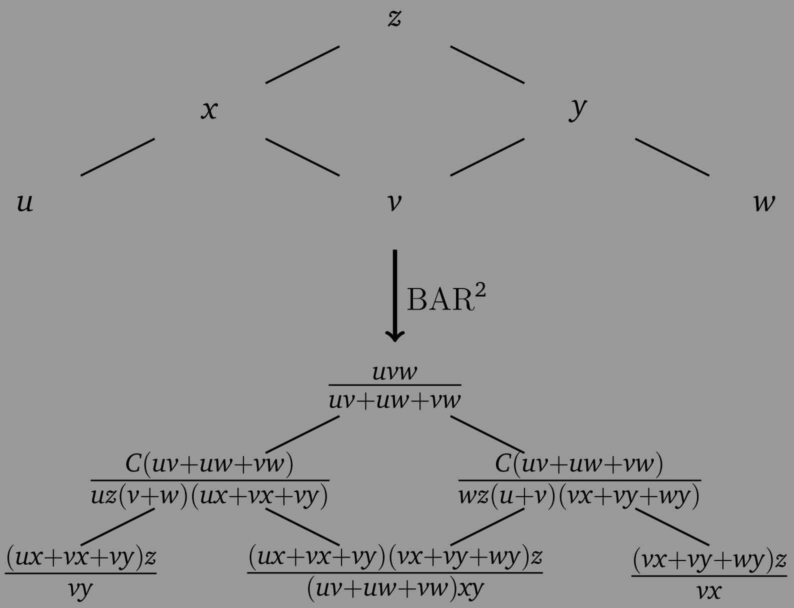


Birational antichain rowmotion



Birational antichain rowmotion



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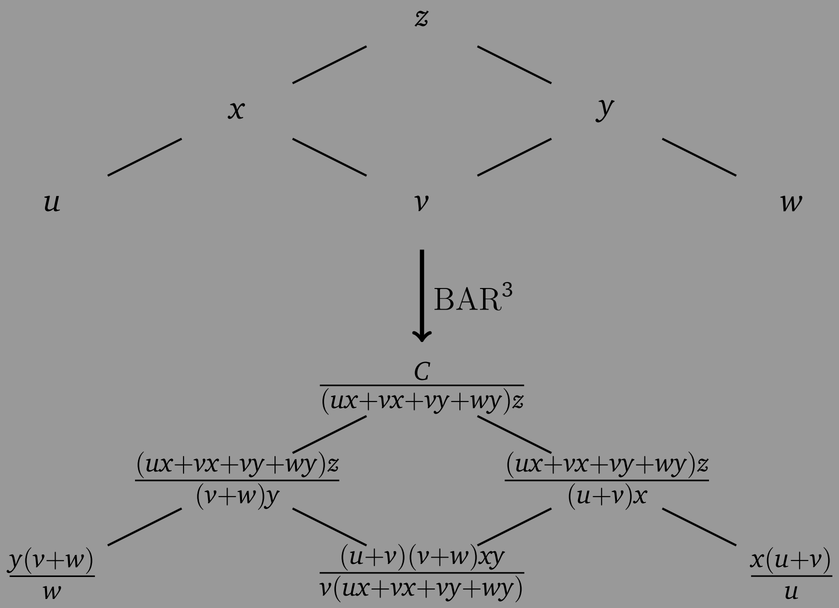

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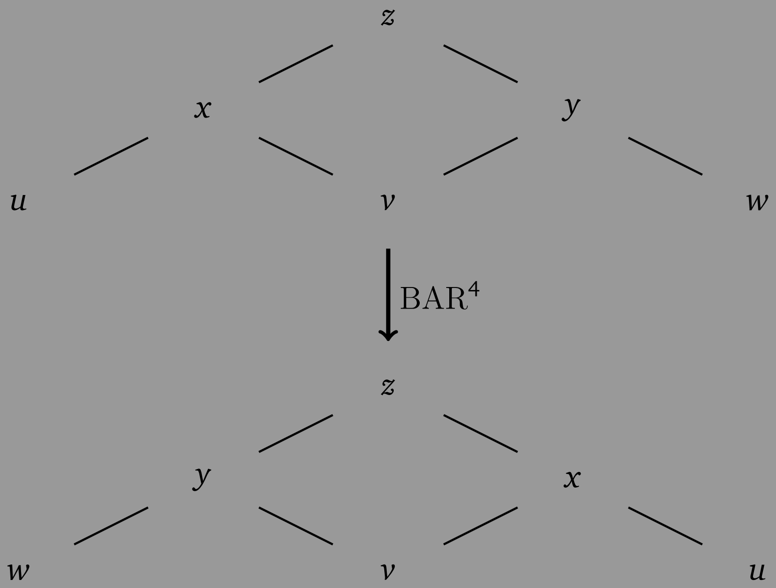

Birational antichain rowmotion

Theorem (Grinberg–Roby 2014)

- *On* A_n, $\mathrm{BAR}^{2(n+1)}$ *is the identity.*
- *On* A_n, BAR^{n+1} *is reflection across the center vertical line.*

Goal

Our goal was to find a lifting of the Lalanne–Kreweras involution on A_n to the piecewise-linear and birational realms, such that the four main properties generalize to these realms.

1. LK^2 is the identity.
2. $\mathrm{LK} \circ \mathrm{Row}_{\mathcal{A}} = \mathrm{Row}_{\mathcal{A}}^{-1} \circ \mathrm{LK}$
3. $\mathrm{card}(A) + \mathrm{card}(\mathrm{LK}(A)) = n$
4. $\mathrm{maj}(A) + \mathrm{maj}(\mathrm{LK}(A)) = n(n+1)$

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It turns out that LK is equivalent to a map called *rowvacuation* on A_n and this allows us to lift LK to the higher realms.

Rowvacuation on antichains

- On any graded poset with rank r, there is an involution *antichain rowvacuation*

  $$\mathrm{Rvac}_{\mathcal{A}} := (\boldsymbol{\tau}_r)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1})\cdots(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1\boldsymbol{\tau}_0)$$

  where again τ_i is the product of all antichain toggles of rank i elements.

- On any graded poset:
  - $\mathrm{Rvac}_{\mathcal{A}}$ is an involution,
  - $\mathrm{Rvac}_{\mathcal{A}} \circ \mathrm{Row}_{\mathcal{A}} = \mathrm{Row}_{\mathcal{A}}^{-1} \circ \mathrm{Rvac}_{\mathcal{A}}$.

Rowvacuation is the Lalanne–Kreweras involution

Theorem (Hopkins–J.)

The Lalanne–Kreweras involution LK *is* $\mathrm{Rvac}_{\mathcal{A}}$ *on* A_n.

Definition (Hopkins–J.)

The *piecewise-linear Lalanne–Kreweras involution* $\mathrm{LK}^{\mathrm{PL}}$ is rowvacuation

$$(\boldsymbol{\tau}_r)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1})\cdots(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1\boldsymbol{\tau}_0)$$

on A_n, where we use piecewise-linear toggles.

Definition (Hopkins–J.)

The *birational Lalanne–Kreweras involution* LK^{B} is rowvacuation

$$(\boldsymbol{\tau}_r)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1})\cdots(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1\boldsymbol{\tau}_0)$$

on A_n, where we use birational toggles.

Birational Lalanne–Kreweras involution

We get the following because it is true for rowvacuation.

Proposition

- LK^{B} *is an involution.*
- $\mathrm{LK}^{\mathrm{B}} \circ \mathrm{BAR} = \mathrm{BAR}^{-1} \circ \mathrm{LK}^{\mathrm{B}}$

Birational Lalanne–Kreweras involution

We get the following because it is true for rowvacuation.

Proposition

- LK^{B} *is an involution.*
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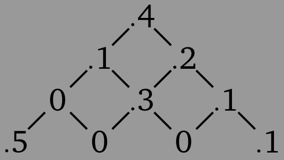
Q: What about the cardinality and major index?

Piecewise-linear "cardinality"

$$\mathrm{card}^{\mathrm{PL}}(g) = \sum_{[i,j]\in \mathsf{A}_n} g([i,j])$$

Example ($n = 4$)

$g =$



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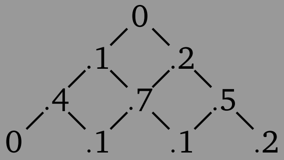

$$\mathrm{card}^{\mathrm{PL}}(g) = .5 + 0 + .1 + 0 + .4 + .3 + .2 + 0 + .1 + .1 = 1.7$$

$$\mathrm{card}^{\mathrm{PL}}\left(\mathrm{LK}^{\mathrm{PL}}(g)\right) = 0 + .4 + .1 + .1 + 0 + .7 + .2 + .1 + .5 + .2 = 2.3$$

Piecewise-linear major index

$$\mathrm{maj}^{\mathrm{PL}}(g) = \sum_{[i,j]\in \mathsf{A}_n} (i+j)g([i,j])$$

Example ($n = 4$)

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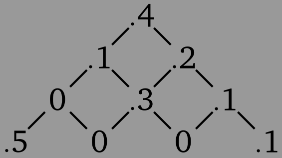
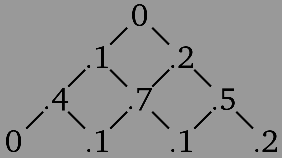


$\mathrm{LK}^{\mathrm{PL}}(g) =$

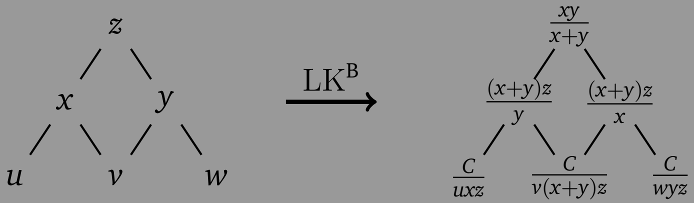


$$2(.5) + 3(0) + 4(.1 + 0) + 5(.4 + .3) + 6(.2 + 0) + 7(.1) + 8(.1) = 7.6$$
$$2(0) + 3(.4) + 4(.1 + .1) + 5(0 + .7) + 6(.2 + .1) + 7(.5) + 8(.2) = 12.4$$

The major indexes add to $20 = 4(4 + 1)$.

Birational cardinality and major index

$$\mathrm{card}^{\mathsf{B}}(g) = \prod_{[i,j]\in \mathsf{A}_n} g([i,j]) \qquad \mathrm{maj}^{\mathsf{B}}(g) = \prod_{[i,j]\in \mathsf{A}_n} g([i,j])^{i+j}$$



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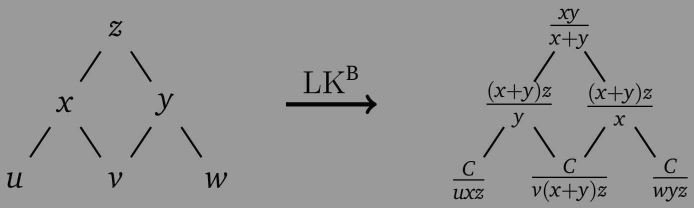

$$\frac{C}{uxz} \cdot \frac{C}{v(x+y)z} \cdot \frac{C}{wyz} \cdot \frac{(x+y)z}{y} \cdot \frac{(x+y)z}{x} \cdot \frac{xy}{x+y} = \frac{C^3}{uvwxyz}$$

$$\left(\frac{C}{uxz}\right)^2 \left(\frac{C}{v(x+y)z}\right)^4 \left(\frac{C}{wyz}\right)^6 \left(\frac{(x+y)z}{y}\right)^3 \left(\frac{(x+y)z}{x}\right)^5 \left(\frac{xy}{x+y}\right)^4 = \frac{C^{12}}{u^2 v^4 w^6 x^3 y^5 z^4}$$

Birational cardinality and major index

Theorem (Hopkins–J.)

For $g \in \mathbb{R}_{\geq 0}^{\mathsf{A}_n}$,

$$\mathrm{card}^{\mathsf{B}}(g)\,\mathrm{card}^{\mathsf{B}}\left(\mathrm{LK}^{\mathsf{B}}(g)\right) = \prod_{[i,j]\in\mathsf{A}_n} g([i,j])\left(\mathrm{LK}^{\mathsf{B}}(g)\right)([i,j]) = C^n$$

$$\mathrm{maj}^{\mathsf{B}}(g)\,\mathrm{maj}^{\mathsf{B}}\left(\mathrm{LK}^{\mathsf{B}}(g)\right) = \prod_{[i,j]\in\mathsf{A}_n} g([i,j])^{i+j}\left(\mathrm{LK}^{\mathsf{B}}(g)\right)([i,j])^{i+j} = C^{n(n+1)}$$

Homomesy

Proposition

Under the action of LK *on* $\mathcal{A}(\mathsf{A}_n)$*,*

1. card *is homomesic with average* $n/2$*,*
2. maj *is homomesic with average* $n(n+1)/2$*.*

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Theorem (Hopkins–J.)

These homomesies lift to the piecewise-linear and birational realms.

More refined homomesies

Theorem

For each $1 \leq i \leq n$*,* $$h_i := \sum_{j=1}^{i} \mathbb{1}_{[j,i]} + \sum_{j=i}^{n} \mathbb{1}_{[i,j]}$$

is 1-mesic under the action of LK *on* $\mathcal{A}(\mathsf{A}_n)$.

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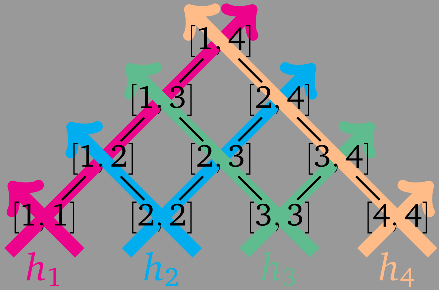

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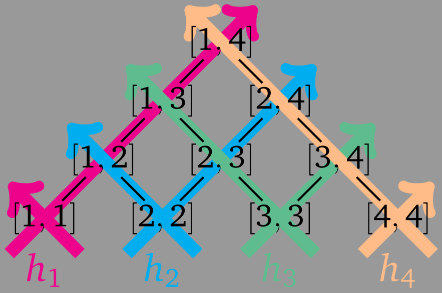
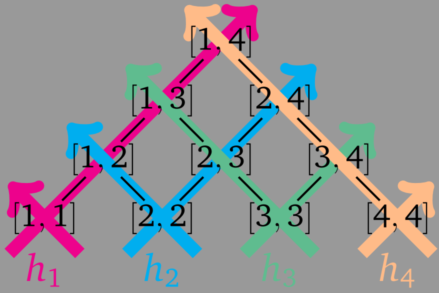


$$\text{card} = \frac{1}{2}(h_1 + h_2 + h_3 + \cdots + h_n)$$

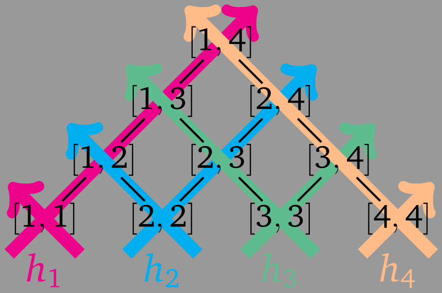
$$\text{maj} = h_1 + 2h_2 + 3h_3 + \cdots + nh_n$$

More refined homomesies



The h_i statistics are the same as those Einstein, Farber, Gunawan, J., Macauley, Propp, Rubinstein-Salzedo proved to be 1-mesic under a product of toggles on noncrossing partitions (2015).

More refined homomesies



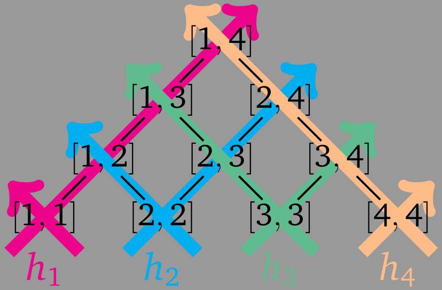
In the *combinatorial* realm, these homomesies are straightforward from the antichain description of LK.

Proposition

If $A = \{[i_1, j_1], \ldots, [i_k, j_k]\}$ *is an antichain of* A_n*, then* $\mathrm{LK}(A) = \{[i'_1, j'_1], \ldots, [i'_{n-k}, j'_{n-k}]\}$ *where*

$$\{i'_1, \ldots, i'_{n-k}\} := [n] \setminus \{j_1, \ldots, j_k\},$$
$$\{j'_1, \ldots, j'_{n-k}\} := [n] \setminus \{i_1, \ldots, i_k\}.$$

More refined homomesies



In the birational realm, the proof of these homomesies uses an embedding (due to Grinberg and Roby) of the labelings of A_n into the product $[n+1] \times [n+1]$ of two chains.

Rowvacuation homomesies yield rowmotion homomesies

Theorem (Hopkins–J.)

Consider a statistic f that is a linear combination of poset-element indicator functions.

1. *If f is homomesic under the action of antichain rowvacuation* $\mathrm{Rvac}_{\mathcal{A}}$*, then f is also homomesic under the action of antichain rowmotion* $\mathrm{Row}_{\mathcal{A}}$.
2. *If f is homomesic under the action of order ideal rowvacuation* $\mathrm{Rvac}_{\mathcal{J}}$*, then f is also homomesic under the action of order ideal rowmotion* $\mathrm{Row}_{\mathcal{J}}$.

The proof is along the same lines as Einstein and Propp's *recombination* argument.

Rowvacuation homomesies yield rowmotion homomesies

Theorem (Armstrong–Stump–Thomas 2011)

Cardinality is homomesic under $\mathrm{Row}_{\mathcal{A}}$ *on* $\mathcal{A}(\mathsf{A}_n)$.

Rowvacuation homomesies yield rowmotion homomesies

Theorem (Armstrong–Stump–Thomas 2011)

Cardinality is homomesic under $\mathrm{Row}_{\mathcal{A}}$ *on* $\mathcal{A}(\mathsf{A}_n)$.

Theorem (Propp 2019)

Major index is homomesic under $\mathrm{Row}_{\mathcal{A}}$ *on* $\mathcal{A}(\mathsf{A}_n)$.

Rowvacuation homomesies yield rowmotion homomesies

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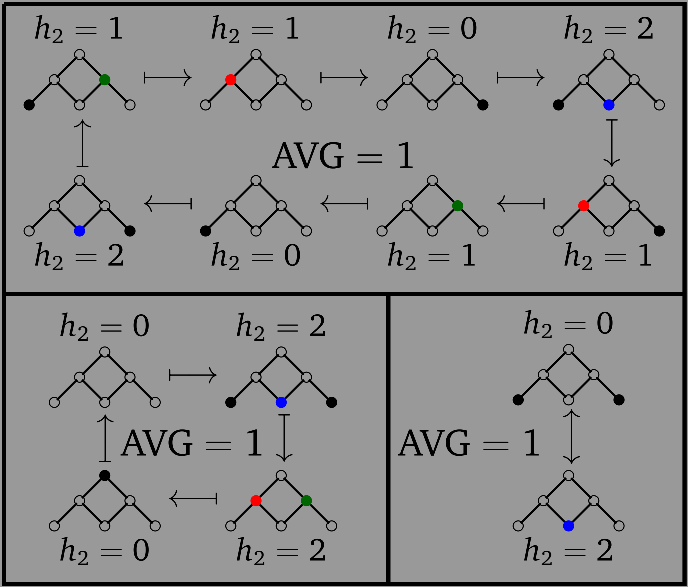
Theorem (Hopkins–J.)

$h_i := \sum_{j=1}^{i} \mathbb{1}_{[j,i]} + \sum_{j=i}^{n} \mathbb{1}_{[i,j]}$ *is homomesic under* $\mathrm{Row}_{\mathcal{A}}$ *on* $\mathcal{A}(\mathsf{A}_n)$.

h_i is homomesic on rowmotion orbits

$$h_2 = \mathbb{1}_{[1,2]} + 2 \cdot \mathbb{1}_{[2,2]} + \mathbb{1}_{[2,3]}$$



Unsolved problem: birational lifting of the OY-invariant?

Oksana Yakimova discovered a statistic $\mathcal{Y} : \mathcal{A}(\mathsf{A}_n) \to \mathbb{Z}_{\geq 0}$ discussed in Panyushev's 2009 paper.

Definition

The *OY-invariant* $\mathcal{Y} : \mathcal{A}(\mathsf{A}_n) \to \mathbb{Z}_{\geq 0}$ is

$$\mathcal{Y}(A) := \sum_{e \in A} \left(|\nabla(F \setminus \{e\})| - |A| + 1 \right)$$

where F is the order filter generated by A and $\nabla(F \setminus \{e\})$ is the set of minimal elements of $F \setminus \{e\}$.

Theorem (Panyushev–Yakimova 2009)

For any $A \in \mathcal{A}(\mathsf{A}_n)$, we have $\mathcal{Y}(\mathrm{Row}_{\mathcal{A}}(A)) = \mathcal{Y}(A) = \mathcal{Y}(\mathrm{LK}(A))$.

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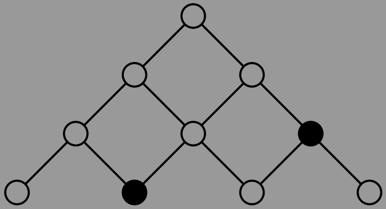
Example



$$\mathcal{Y}(A) = ? + ?$$

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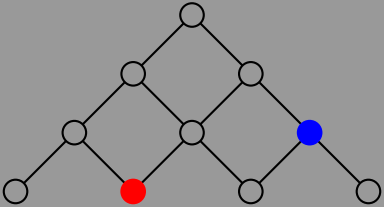

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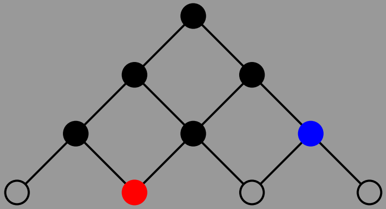
Example



$$\mathcal{Y}(A) = ? + ?$$

Unsolved problem: birational lifting of the OY-invariant?

Definition

The *OY-invariant* $\mathcal{Y} : \mathcal{A}(\mathsf{A}_n) \to \mathbb{Z}_{\geq 0}$ is

$$\mathcal{Y}(A) := \sum_{e \in A} \left(|\nabla(F \setminus \{e\})| - |A| + 1 \right)$$

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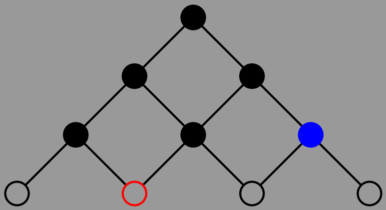

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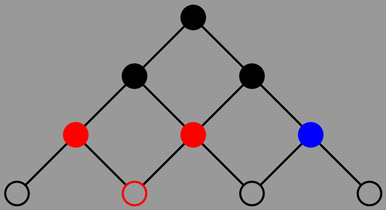

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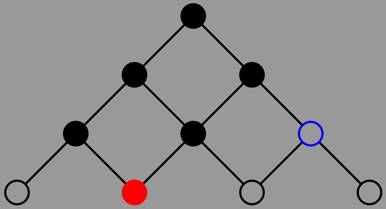

$$\mathcal{Y}(A) = \mathbf{2} + \mathbf{0}$$

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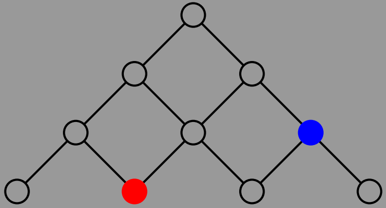
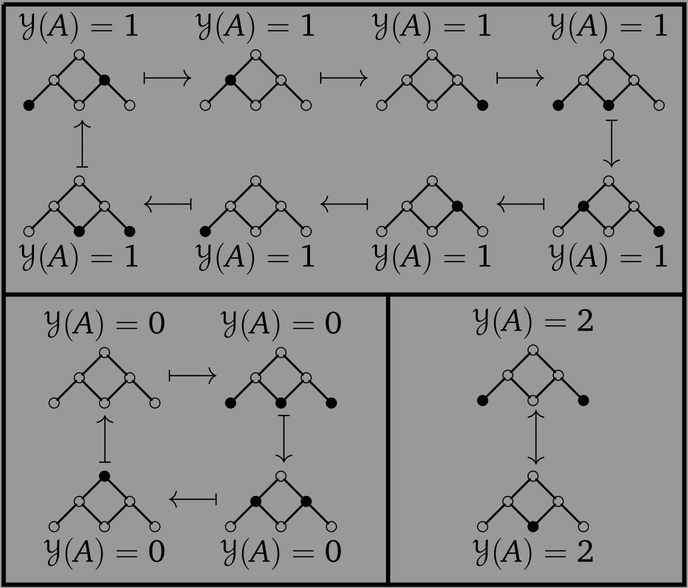


$$\mathcal{Y}(A) = \mathbf{2} + \mathbf{0} = 2$$

Unsolved problem: birational lifting of the OY-invariant?



Unsolved problem: birational lifting of the OY-invariant?

Question: How could we lift the OY-invariant to the higher realms if there is no "antichain" to sum over?

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where

$$\mathcal{Y}_{[i,j]}(A) := \begin{cases} |\nabla(F \setminus \{e\})| - |A| + 1 & \text{if } [i,j] \in A, \\ 0 & \text{if } [i,j] \notin A. \end{cases}$$

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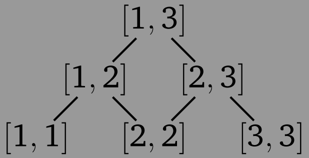
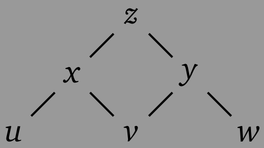
Find an equivalent description of $\mathcal{Y}_{[i,j]}$ that doesn't ask if $[i,j]$ is in the antichain.

Unsolved problem: birational lifting of the OY-invariant?

We have found a way to lift $\mathcal{Y}$ to the birational realm! But we do **not** know how to prove $\mathcal{Y}^{\mathrm{B}}$ is invariant under BAR or LK^{B}.

Example (A_3)





$$\begin{aligned}\mathcal{Y}^{\mathrm{B}}(g) &= \mathcal{Y}^{\mathrm{B}}_{[1,1]}(g)\mathcal{Y}^{\mathrm{B}}_{[2,2]}(g)\mathcal{Y}^{\mathrm{B}}_{[3,3]}(g)\mathcal{Y}^{\mathrm{B}}_{[1,2]}(g)\mathcal{Y}^{\mathrm{B}}_{[2,3]}(g)\mathcal{Y}^{\mathrm{B}}_{[1,3]}(g)\\ &= \frac{u+v}{v}\cdot\frac{u+v}{u}\cdot\frac{v+w}{w}\cdot\frac{v+w}{v}\cdot\frac{vx+vy+wy}{(v+w)y}\cdot\frac{ux+vx+vy}{(u+v)x}\cdot 1\\ &= \frac{(ux+vx+vy)(vx+vy+wy)(u+v)(v+w)}{uv^2wxy}.\end{aligned}$$

Thank You!

Fixed points

Q: How many antichains $A \in \mathcal{A}(\mathsf{A}_n)$ satisfy $A = \mathrm{LK}(A)$?

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It is easy to see that these are exactly the antichains $\{[i_1, j_1], [i_2, j_2], \cdots, [i_{n/2}, j_{n/2}]\}$ in which each of $1, 2, \ldots, n$ appear exactly once among $i_1, i_2, \cdots, i_{n/2}, j_1, j_2, \cdots, j_{n/2}$.

Example (in A_8)

$A = \{[1, 2], [3, 5], [4, 7], [6, 8]\}$

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These correspond to standard Young tableaux of the two-rowed rectangle with $n/2$ columns.

1	3	4	6
2	5	7	8

$$\#\{A \in \mathcal{A}(\mathsf{A}_n) : A = \mathrm{LK}(A)\} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathrm{Cat}\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

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Q: How many antichains $A \in \mathcal{A}(\mathsf{A}_n)$ are fixed under $\mathrm{LK} \circ \mathrm{Row}_{\mathcal{A}}$?

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Proposition (Hopkins–J.)

$$\#\{A \in \mathcal{A}(\mathsf{A}_n) : A = \mathrm{LK}\left(\mathrm{Row}_{\mathcal{A}}(A)\right)\} = \binom{n+1}{\lfloor (n+1)/2 \rfloor}$$

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$\binom{n+1}{\lfloor (n+1)/2 \rfloor}$ is also the number of antichains that are symmetric across the center vertical line. We showed that $\mathrm{LK} \circ \mathrm{Row}_{\mathcal{A}}$ is conjugate to $\mathrm{flip} = \mathrm{Row}_{\mathcal{A}}^{n+1}$ in the toggle group.