Promotion via representations of quivers

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BIRS Workshop: Dynamical Algebraic Combinatorics

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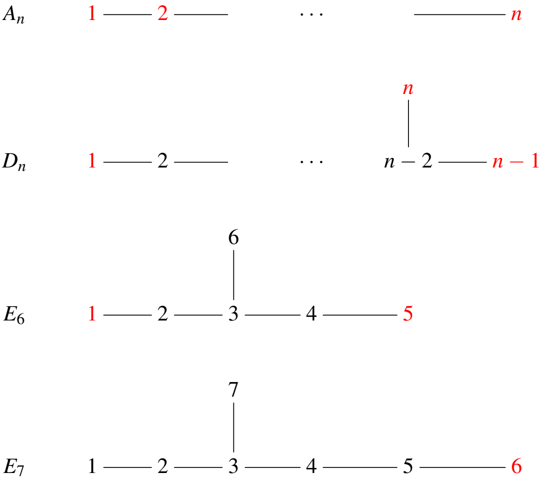
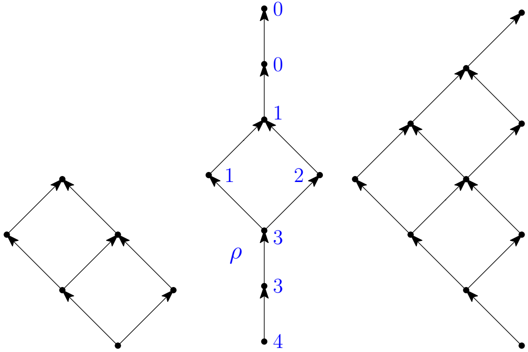
Minuscule posets

The minuscule posets are defined by choosing a simply-laced Dynkin diagram and a **minuscule vertex** m.



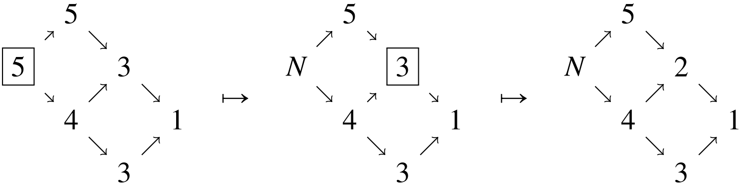
Minuscule posets



$[3] \times [2]$

A **reverse plane partition** on a poset P is an order-reversing map $\rho : \mathsf{P} \to \{0, 1, \ldots, N\}$ for some N.

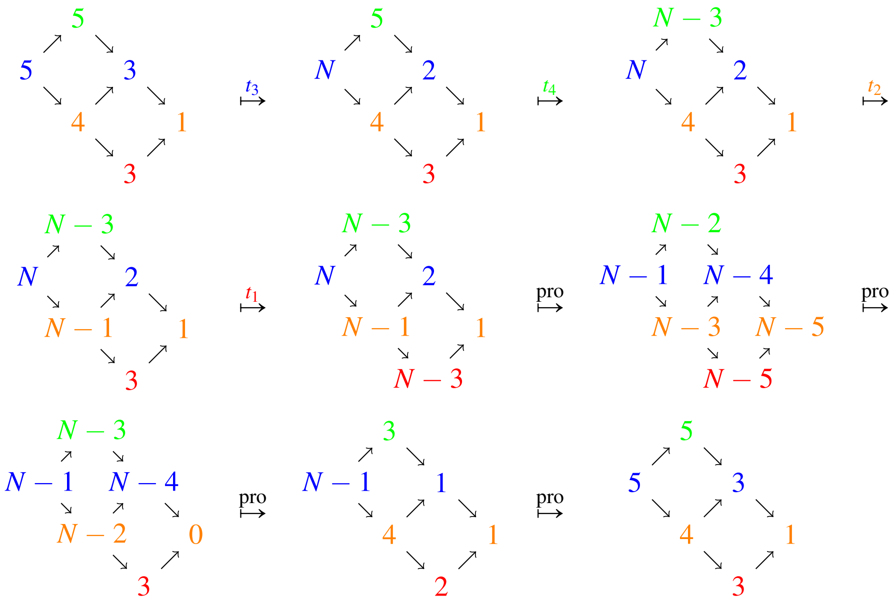
Toggling on reverse plane partitions



Given $\mathsf{x} \in \mathsf{P}$, one can **toggle** a reverse plane partition ρ as follows to produce a new reverse plane partition $t_{\mathsf{x}}\rho$.

$$t_{\mathsf{x}}\rho(\mathsf{y}) = \begin{cases} \max\limits_{\mathsf{y}<\mathsf{y}_1} \rho(\mathsf{y}_1) + \min\limits_{\mathsf{y}_2<\mathsf{y}} \rho(\mathsf{y}_2) - \rho(\mathsf{y}) & : \text{ if } \mathsf{y} = \mathsf{x} \\ \rho(\mathsf{y}) & : \text{ if } \mathsf{y} \neq \mathsf{x}, \end{cases}$$

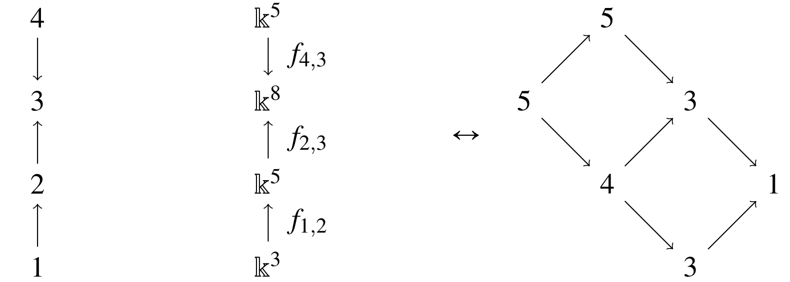
Periodicity for promotion pro $= t_1 t_2 t_4 t_3$



Theorem (Grinberg–Roby '15, Musiker–Roby '18)

Promotion on reverse plane partitions of $P = [a] \times [b]$ has order $a + b$.

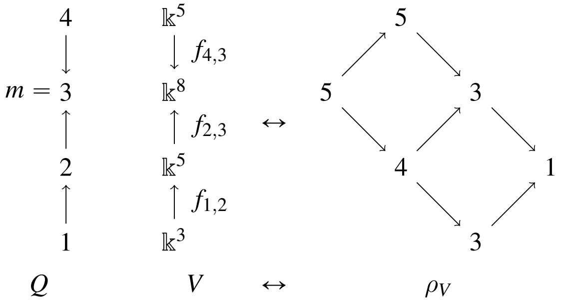
- Proved using a birational version of the promotion operator and then "tropicalizing".
- We have a new proof of periodicity by showing that reverse plane partitions are equivalent to certain representations of quivers.



Q **quiver** | V **quiver representation** with dimension vector $\mathbf{dim}(V) = 3585$ | $\leftrightarrow$ | ρ_V

Theorem (G.–Patrias–Thomas '18)

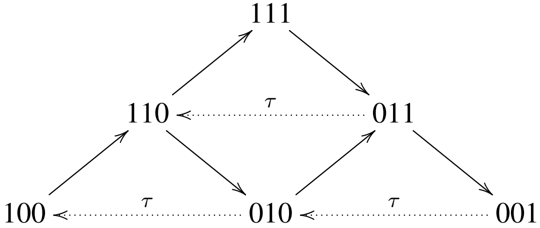
Let Q be a Dynkin quiver, m a minuscule vertex of Q, and $\mathsf{P}_{Q,m}$ the associated minuscule poset. There is a bijection between representations of Q all of whose indecomposable summands are supported at m and reverse plane partitions of $\mathsf{P}_{Q,m}$.



- To obtain ρ_V from V, one calculates the Jordan blocks of a generic nilpotent endomorphism of V.
- In the example, the Jordan blocks are ((3), (4,1), (5,3), (5)).

$$\begin{array}{ccc} Q & \rightsquigarrow & \Gamma(Q) \\ \text{quiver} & & \textbf{Auslander–Reiten quiver} \end{array}$$

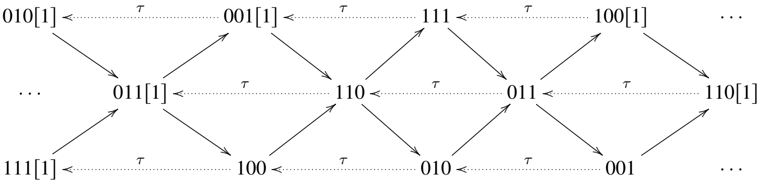
$$\begin{array}{ccc} \{\text{vertices of } \Gamma(Q)\} & \leftrightarrow & \{\text{indecomposable representations of } Q\} \\ \{\text{arrows of } \Gamma(Q)\} & \leftrightarrow & \{\text{irreducible morphisms up to rescaling}\} \end{array}$$



$\Gamma(Q)$ where $Q = 1 \leftarrow 2 \leftarrow 3$

The functor τ is known as the **Auslander–Reiten translation**.

The category of representations of Q sits inside the *root category* $\mathcal{R}_Q$.



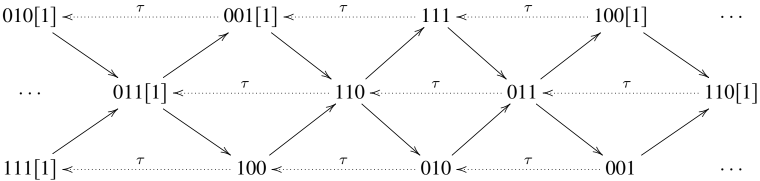
$\Gamma(\mathcal{R}_Q)$where $Q = 1 \leftarrow 2 \leftarrow 3$

Lemma

The isomorphism classes of indecomposable objects of $\mathcal{R}_Q$ are given by

$$\{X, X[1] \mid X \in ind(rep(Q))\}.$$

The category of representations of Q sits inside the *root category* $\mathcal{R}_Q$.



$\Gamma(\mathcal{R}_Q)$ where $Q = 1 \leftarrow 2 \leftarrow 3$

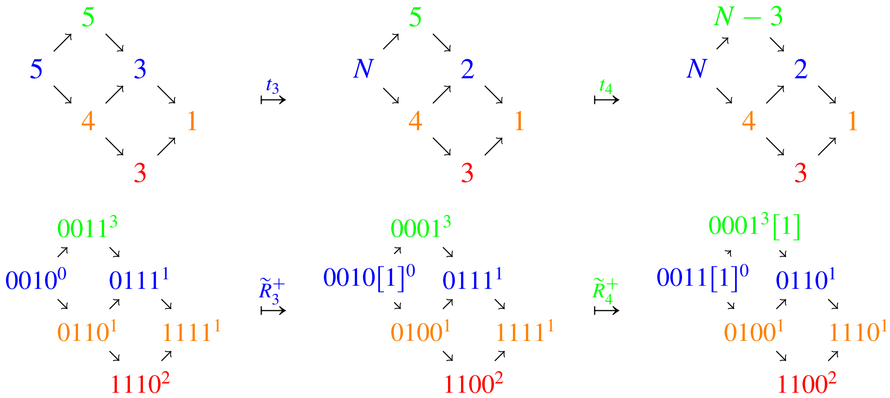
Lemma

Let Q be a Dynkin quiver. For any object $X \in \mathcal{R}_Q$,

$$\tau^h(X) = X$$

where h is the Coxeter number of the Coxeter group associated with Q. When Q is a type A Dynkin quiver, $h = |\{\text{vertices of } Q\}| + 1$.

Promotion is compatible with τ

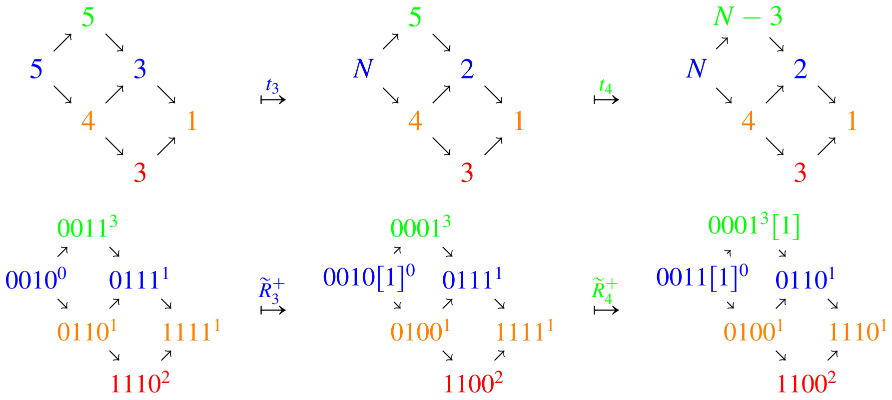


The reflection functor $\widetilde{R}_i^+$ acts on dimension vectors as follows

$$\begin{aligned} \widetilde{R}_i^+ : \mathbb{Z}^{|Q_0|} &\longrightarrow \mathbb{Z}^{|Q_0|} \\ (x_1, \ldots, x_i, \ldots, x_n) &\longmapsto (x_1, \ldots, -x_i + \textstyle\sum_{k \,—\, i} x_k, \ldots, x_n) \\ \mathbf{dim}(X) &\longmapsto \mathbf{dim}(\widetilde{R}_i^+(X)) \end{aligned}$$

if $X \in \mathrm{rep}(Q)$ has no summands of $S(i)$.

Promotion is compatible with τ



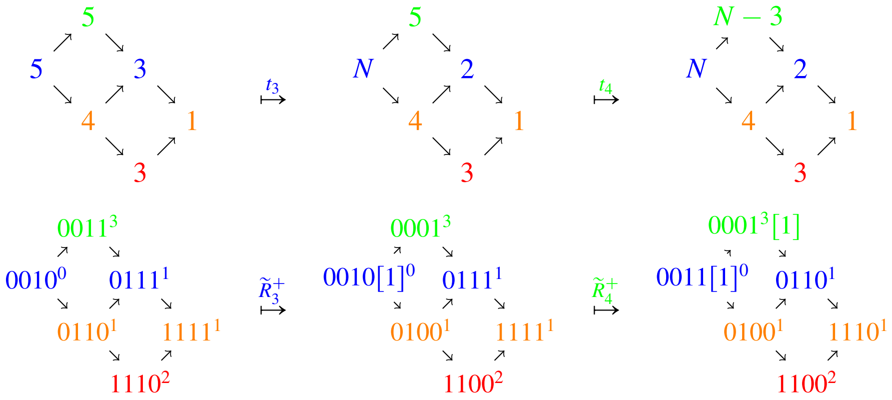
Theorem (G.–Patrias–Thomas '18)

For any $X \in \mathcal{C}_{Q,m} \subset \mathcal{R}_Q$, *one has* $t_i \rho(X) = \rho(\widetilde{R}_i^+(X))$.

Lemma (Gabriel '80)

For any $X \in \mathcal{R}_Q$, $\tau(X) \simeq \widetilde{R}_{i_n}^+ \cdots \widetilde{R}_{i_1}^+(X)$.

Promotion is compatible with τ



Corollary

For any $X \in \mathcal{C}_{Q,m} \subset \mathcal{R}_Q$, *one has*

$$\rho(\tau(X)) = \rho(\widetilde{R}_{i_n}^+ \cdots \widetilde{R}_{i_1}^+(X)) = t_{i_n} \cdots t_{i_1} \rho(X) = pro \rho(X).$$

In particular, $pro^h \rho(X) = \rho(\tau^h(X)) = \rho(X)$.

Thanks!