Knot Floer Homology as immersed curves

(work in progress)

Tuesday, June 2, 2020 11:27 PM

Knot Floer homology (Ozsváth-Szabó, Rasmussen)

$\boxed{\mathbb{F} = \text{field}}$

$\left\{\begin{array}{c} K \subset Y^3 \\ \text{nullhomologous} \\ \text{knot} \end{array}\right\} \rightsquigarrow$

① $CFK^-(Y,K)$ bigraded chain cpx over $\mathbb{F}[U,V] =: R$ (up to chain htpy equiv)

② Isomorphism $f: HF^-(Y) \to HF^-(Y)$ "flip map"

- computed from doubly pointed Heegaard diagram
- forgetting either basepoint gives Heegaard diag. for Y
- sequence of Heegaard moves taking one basepoint to the other gives rise to "flip map"

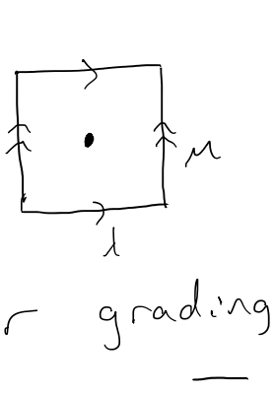
Goal: Interpret the data on the right as a geometric object

Thm (H.) $CFK^-(Y,K)$ w/ flip map f can be represented by $(\Gamma, \underline{b})$ where

- Γ is a (bigraded) immersed multicurve in the marked torus
- $\underline{b}$ is a bounding cochain

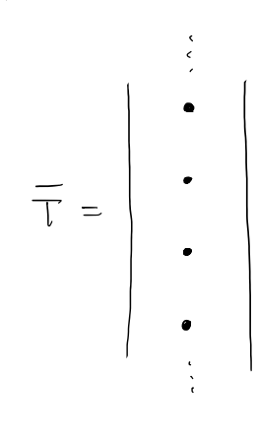
Remarks:

- marked torus = T^2 with one marked point and chosen basis $\{\mu, \lambda\}$ for $H_1(T^2)$ $=: T$

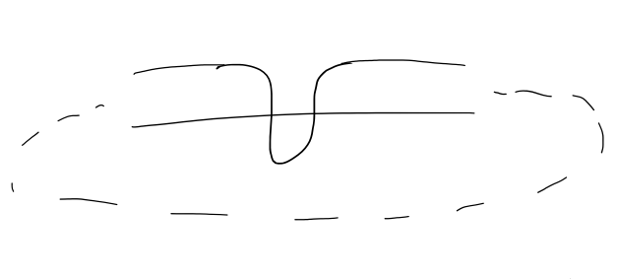
$T =$ 

- Alexander grading specifies a lift of Γ to the cylinder $\overline{T} := \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ (with marked points at $\{0\} \times \mathbb{Z}$)

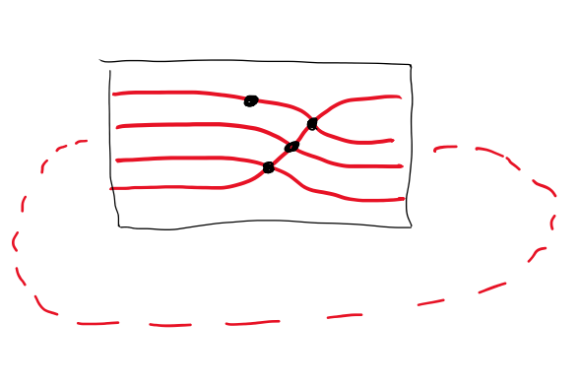
  we will generally work in $\overline{T}$

$\overline{T} =$ 

- Maslov grading – standard notion of grading on immersed Lagrangian
- $\underline{b}$ is a linear combination (over $\mathbb{F}$) of $\{\text{self intersection pts of } \Gamma\} \cup \{\text{a chosen pt on each component of } \Gamma\}$ satisfying a "no monogons" condition
- We may assume:
  - Γ is in minimal position, except ∄ immersed annuli



  - $\underline{b}$ contains no points of negative index
  - The only zero index points in $\underline{b}$ are of the following form



(Note: index 0 part of $\underline{b}$ $\updownarrow$ local system on each curve)

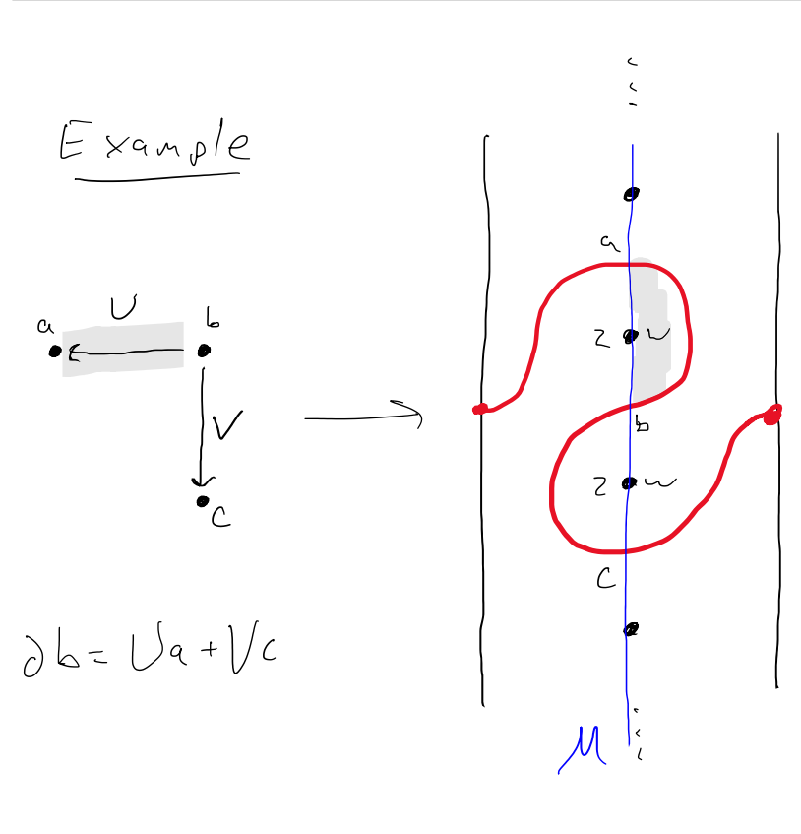
- With these assumptions
  - Γ is an invariant of (Y,K) up to homotopy
  - index 0 part of $\underline{b}$ is an invariant
  - $\underline{b}$ only invariant up to some equivalence

    Q: "normal form" for $\underline{b}$

Thm 1′ Any chain htpy equiv. class of bigraded complex over R can be represented by $(\Gamma, \underline{b})$ where

- Γ is a collection of immersed curves/arcs in strip $S = [-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R}$ (marked points at $\{0\} \times \mathbb{Z}$) w/ $\partial\Gamma \subset \partial S$
- $\underline{b}$ is a bounding cochain

Example



$\partial b = Ua + Vc$

To recover complex from curves:

- add basepoints left/right of each marked point
- take Floer homology with vertical line μ through marked points

For CFK^-, we can identify sides of S to get $\overline{T}$ (flip map tells us how to match up loose ends)

Thm 2 (Surgery formula)

$HF^-(Y_{p/q}(K))$ is isomorphic to the Floer homology (in the marked torus T) of $(\Gamma, \underline{b})$ with a line of slope p/q

Context

If we restrict to $Y = S^3$ and consider the $UV=0$ quotient of CFK^-, the analogous statements follow from earlier work with Rasmussen and Watson

$$\begin{tikzcd}
CFK(S^3,K)|_{UV=0} \arrow[r, leftrightarrow, "\text{Lipshitz-Ozsváth-Thurston}"] \arrow[dr, dashed, bend left, "KWZ"] \arrow[dr] & \widehat{CFD}(S^3 \setminus nbhd(K)) \arrow[d, leftrightarrow, "HRW"] \\
 & \widehat{HF}(S^3 \setminus nbhd(K))
\end{tikzcd}$$

chain cpx over $\hat{R} := R/_{UV=0}$

type D structure over torus algebra

immersed curves with local systems in punctured torus

- The HRW arrow is a special case of a more general correspondence.

Thm: (Haiden-Katzarkov-Kontsevich)

For S a "surface with marked boundary"

$\left\{\begin{array}{c} \text{Type D structures} \\ \text{over } A(S) \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \text{immersed curves in } S \\ \text{with local systems} \end{array}\right\}$

$\| $ Fuk(S) ← algebra associated to

- HRW gave a more constructive proof, but restricted to compact part of Fuk(S) and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$
- Kotelskiy-Watson-Zibrowius extended this constructive proof to recover result above

$$\begin{tikzcd}
CFK^-(Y,K) \arrow[r, dashed] \arrow[dr] & ?? \arrow[d, dashed, "?"] \\
 & (\Gamma, \underline{b})
\end{tikzcd}$$

← bordered HF^-

bigraded cpx over R

immersed curves with bounding cochain in marked torus

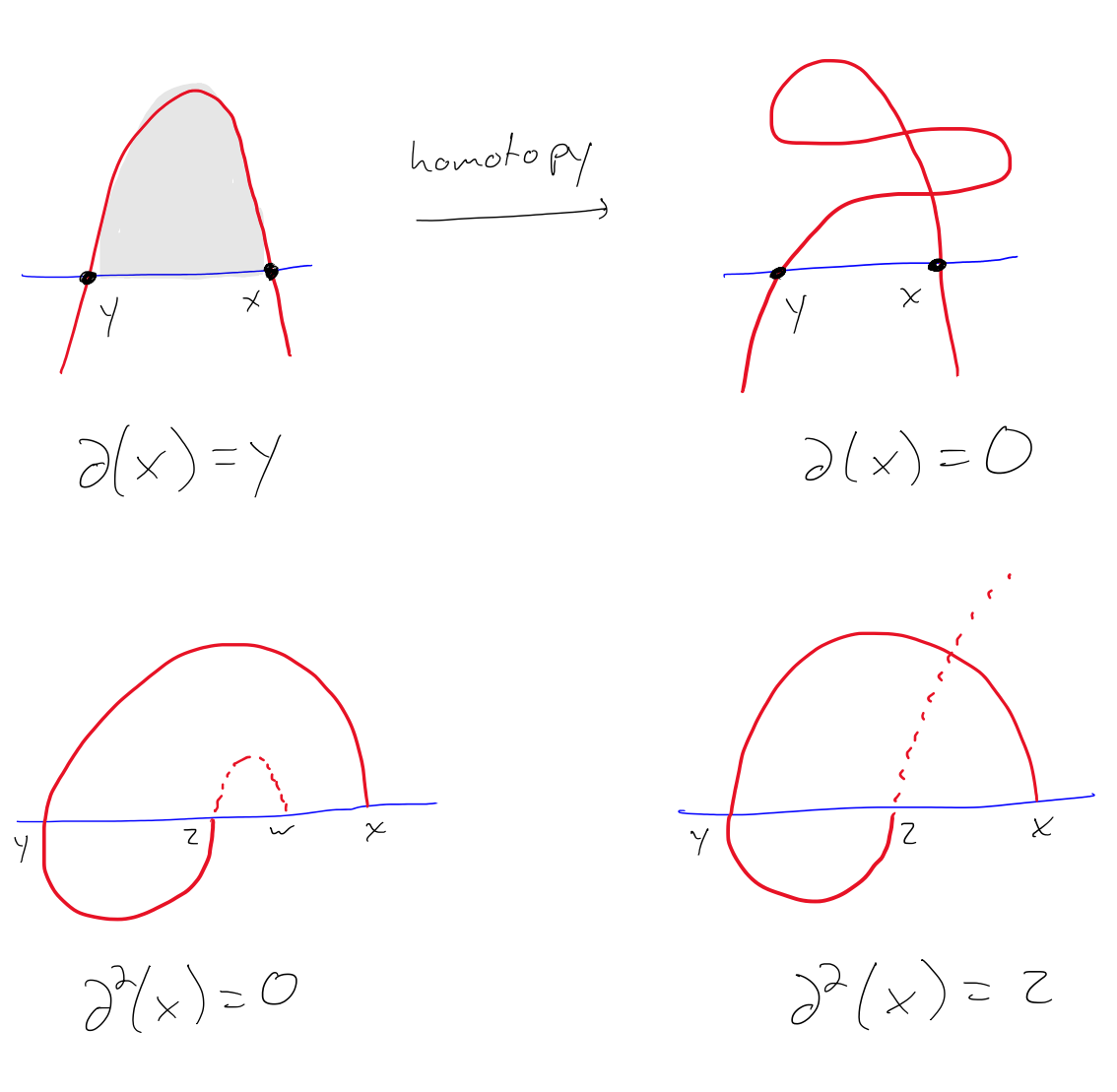
Conjecture: $(\Gamma, \underline{b})$ also represents bordered HF^- of $Y \setminus K$ and arbitrary pairing given by Floer homology in T

Take away:

- difference between "minus" and "hat" invariants is difference between Floer homology of marked vs. punctured surface
- Extra info contained in "minus" is bounding cochain

Immersed curves and bounding cochains

Consider Floer homology of two curves



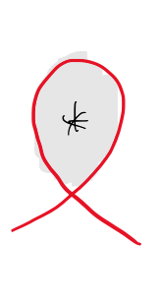
Problem: monogons are bad!

Solution 1: Don't allow monogons

[Def An immersed curve is unobstructed if it bounds no monogons]

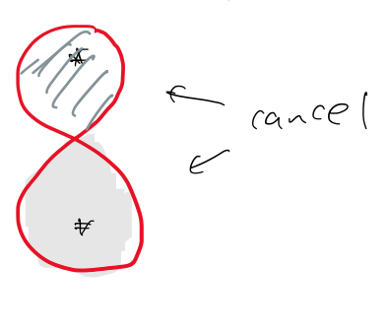
This works fine in punctured torus...

can homotope away monogons unless they enclose puncture



But these count in marked torus can't remove these

Better: require signed count of monogons = 0



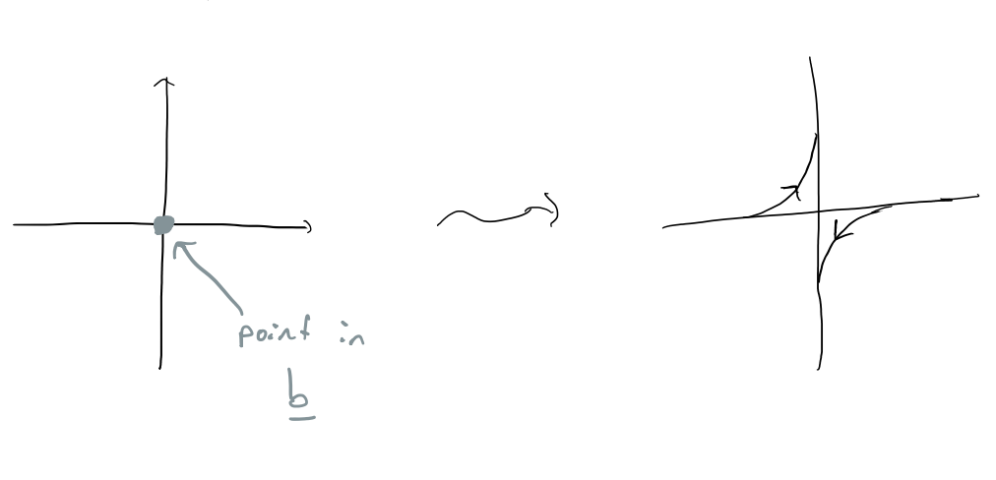
Solution 2:

ⓐ add bounding cochain $\underline{b} \in CF^{odd}(L,L)$

generated by – self int. pts of Γ – one point per component

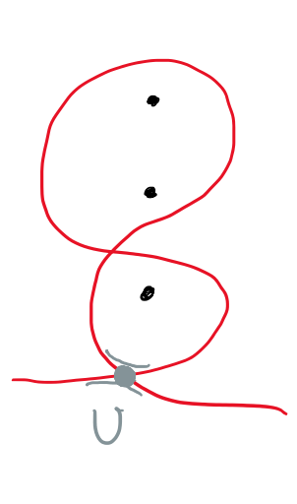
ⓑ Modify the Floer differential "allow left turns at int. pts. in $\underline{b}$"

- ∂ counts $(k+2)$-gons with k corners in $\underline{b}$
- Equivalently, replace Γ with immersed train track



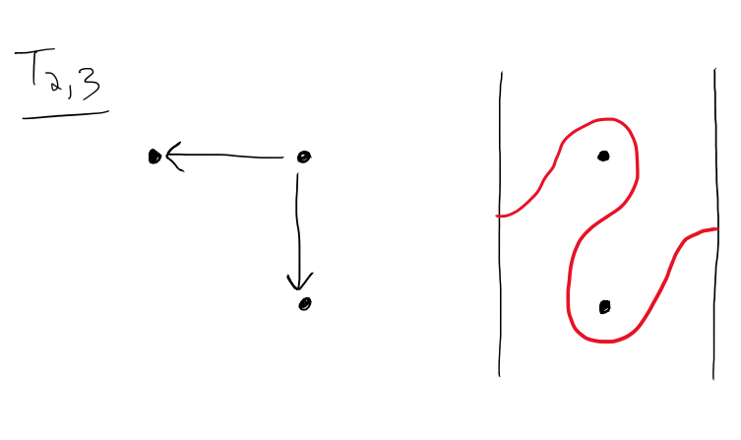
Def $\underline{b}$ is a bounding cochain if the (modified) signed count of monogons is zero.

Note: turning arcs at pt in $\underline{b}$ weighted by $c \in \mathbb{F}$. also weighted by power of U (power determined by gradings)



Examples

$T_{2,3}$



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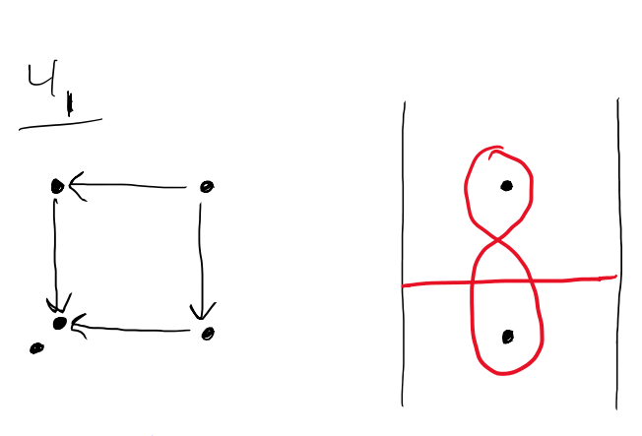
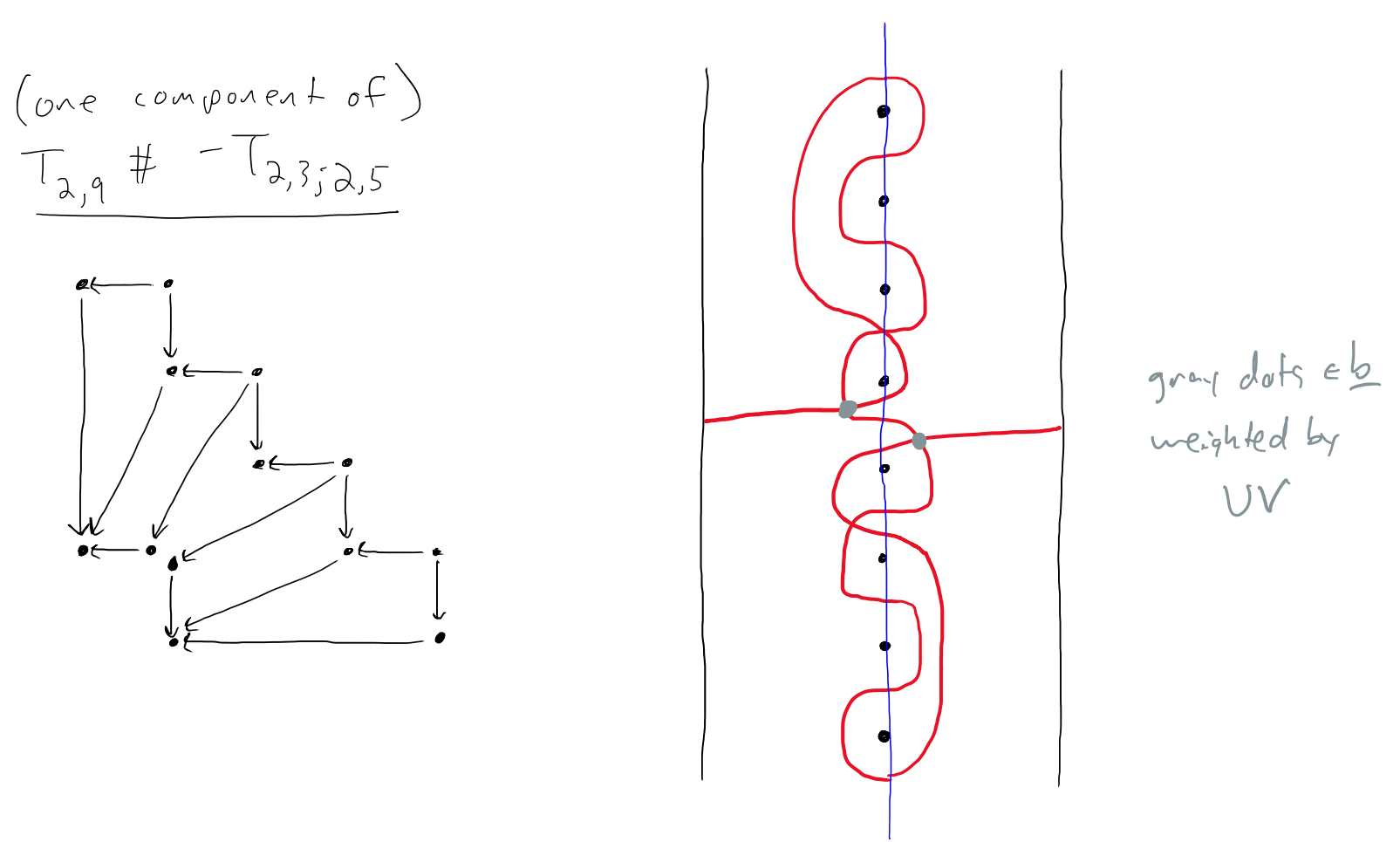
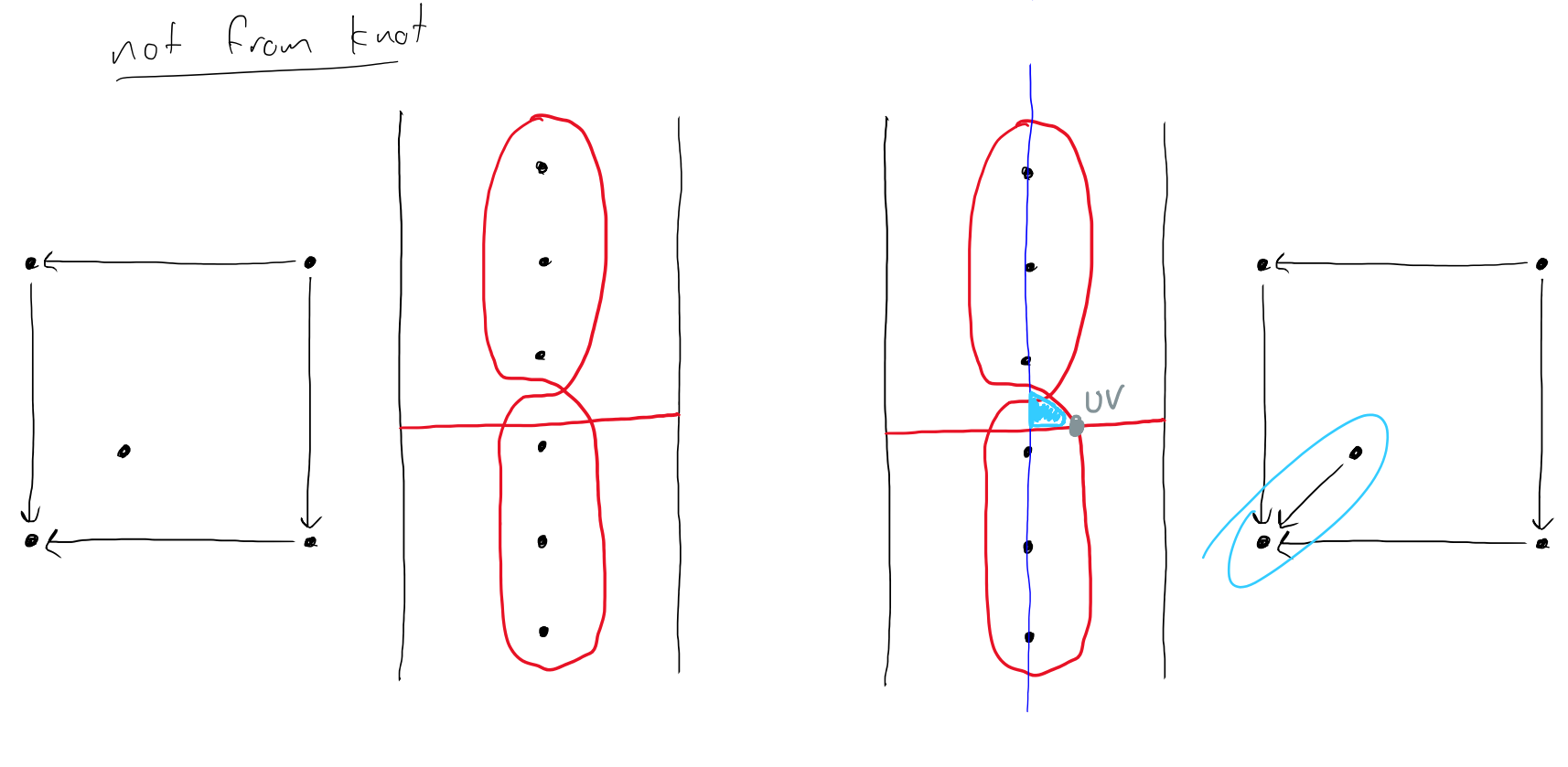


(one component of) $T_{2,9} \# -T_{2,3;2,5}$



not from knot



Aside: For $K \subset S^3$, the multicurve $\Gamma = \{\gamma_0, \ldots, \gamma_n\}$ intersects $x = \pm\frac{1}{2}$ exacly once we assume this happens on γ_0

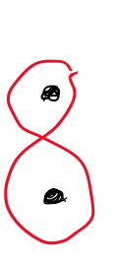
- γ_0 is a concordance invariant
- other invariants $(\tau, \varepsilon, \phi_i, \ldots)$ easily extracted from γ_0

Applications

① Often there is only one choice of $\underline{b}$ given Γ $\Rightarrow CFK^-$ determined by $CFK|_{UV=0}$ ← computable

e.g. If Γ embedded (L-space knots, cables of L-space knots)

Claim We can choose $\underline{b}$ to avoid any points on a simple figure 8



Corollary: If one component of Γ is embedded and rest are figure eights, CFK^- is determined by $CFK|_{UV=0}$

↳ e.g.
- thin knots
- all but one prime knot with ≤ 15 crossings

② Obstructing cosmetic surgeries

Cosmetic surgery conjecture: For $K \subset S^3$ nontrivial, if $S^3_r(K) = S^3_{r'}(K)$ then $r = r'$

Some results: (suppose $r \neq r'$)

$H_1 \Rightarrow$ same numerator $r = \frac{p}{q}$, $r' = \frac{p}{q'}$ $(p > 0)$

(Ozsváth-Szabó + Wu): q and q' have opposite sign

↖ used rk $\widehat{HF}$

(Ni-Wu): used d-invts

- $\tau(K) = 0$ ← in fact, γ_0 trivial
- $q' = -q$
- $q^2 \equiv -1 \pmod{p}$

Thm (H.) If $S^3_{p/q}(K) \cong S^3_{-p/q}(K)$ then

- $p = 1$ or $p = 2$
- If $p = 2$, then $q = 1$ and $g(K) = 2$
- If $p = 1$, then $q \leq \frac{t + 2g}{g(g-1)}$

  where t is Heegaard Floer thickness of K
- other constraints on CFK

Cor: The conjecture holds for
- thin knots with $g \neq 2$
- all knots ≤ 16 crossings

Also holds for
- 2-bridge knots (Ichihara-Jong-Mattman-Saito)
- non-prime knots (Tao)

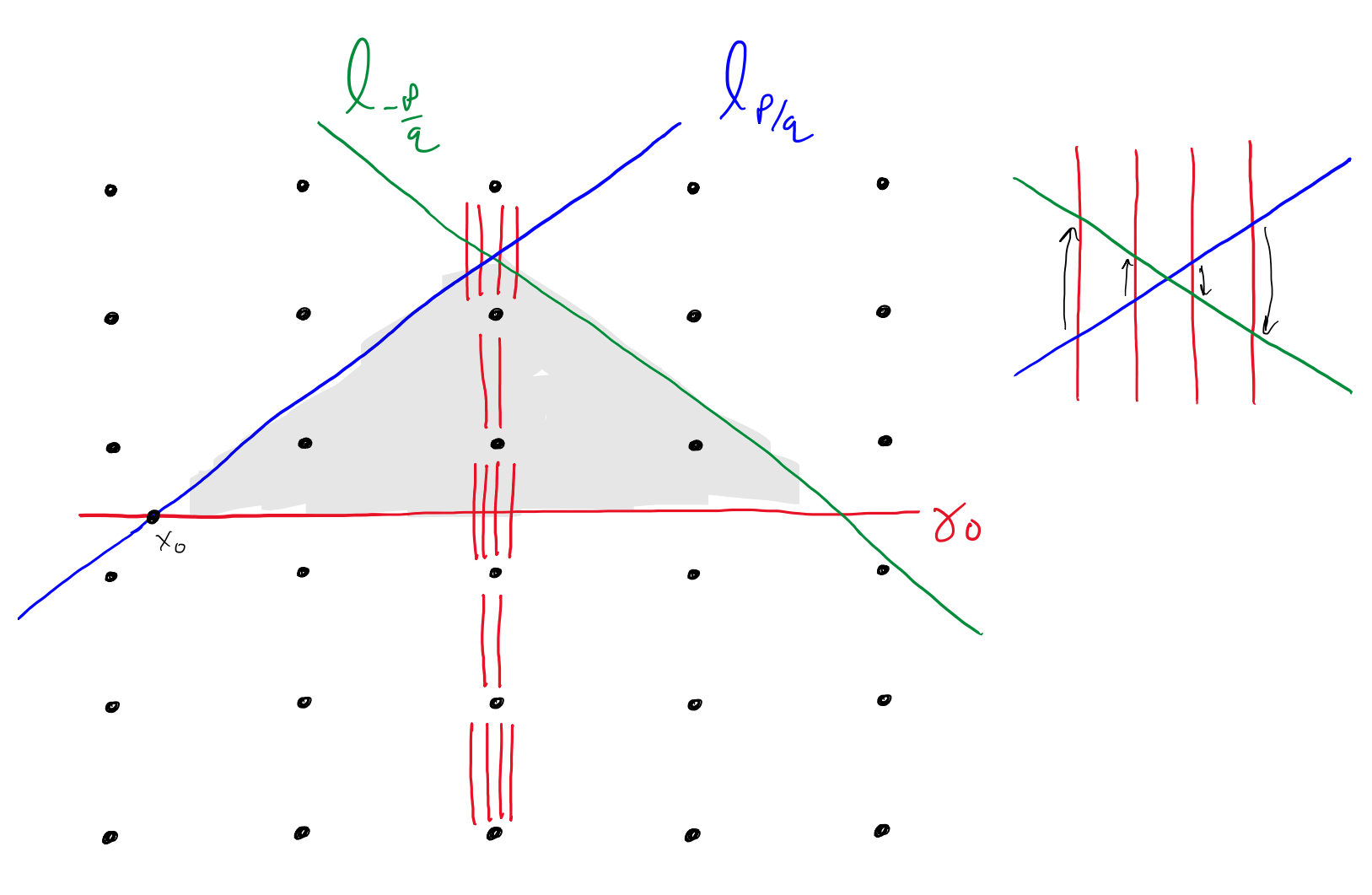
Idea of proof

Let Γ be curves corresponding to CFK(K)

May assume γ_0 is horizontal

All other curves in nbhd of μ

assume vertical outside nbhd of marked points



∃ obvious map $\Gamma \cap l_{p/q} \longrightarrow \Gamma \cap l_{-p/q}$

This gives (ungraded) isomorphism

$\phi: \widehat{HF}(S^3_{p/q}(K)) \longrightarrow \widehat{HF}(S^3_{-p/q}(K))$

Q: What does ϕ do to gradings?

Key Lemma: For $x \neq x_0$, $gr(\phi(x)) - gr(x)$

$$gr(\phi(x)) - gr(x) = 1 - 2(\#\text{ punctures in shaded triangle})$$

Key observation: grading change is

$\left\{\begin{array}{ll} +1 & \text{if } x \text{ on vertical segment at height } 0 \\ \leq -1 & \text{otherwise} \end{array}\right.$

Thm (Wang): Conjecture holds if $g(K) = 1$

reproof: If $g = 1$, all vertical segments at height 0 $\Rightarrow$ net increase in gradings under ϕ //

On the other hand, must have at least half of all vertical segments at height 0

Large $g(K)$ and/or q $\Rightarrow$ ∃ generator with large grading drop

$\Rightarrow$ ∃ large chain of generators with +1 grading increase

$\Rightarrow$ large thickness