A Nonlinear Plancherel Theorem and Application to Global Well-Posedness for the Defocusing Davey-Stewartson Equation (and the Inverse Boundary Value Problem of Calderón)

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BIRS February 3, 2020

Overview

The Davey-Stewartson Equations

The Davey-Stewartson family of equations were initially introduced in the study of water waves (they model the evolution of weakly nonlinear surface water waves in 2+1 dimensions, travelling principally in one direction). They also arise in the context of ferromagnetism, plasma physics, and nonlinear optics.

LWP for the L^2 critical case and GWP for small initial data (using dispersive methods):

- Ghidaglia and Saut (1990)
- Linares and Ponce (1993)
- Hayashi and Saut (1995)

In this talk we consider one special member of this family: defocusing DSII.

The Defocusing DSII Equations

Defocusing DSII:

$$\begin{cases} i\partial_t q + 2(\bar{\partial}^2 + \partial^2)q + q(g + \overline{g}) = 0 \\ \bar{\partial} g + \partial(|q|^2) = 0 \\ q(0,z) = q_0(z). \end{cases} \tag{1}$$

This model is completely integrable and can be solved by the Inverse-Scattering method.

Notation:

$$z = x_1 + ix_2; \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

- Perry (2014) - GWP for general $q_0 \in H^{1,1}$ using Inverse-Scattering method
- This talk: GWP for q_0 in L^2 (mass critical case), via a Plancherel Theorem for the Scattering Transform.

The Scattering Transform

Lax pair for defocusing DSII: $L_t = [L, A]$, where

$$L : \begin{cases} \bar{\partial} m^1 & = q m^2 \\ (\partial + ik) m^2 & = \overline{q} m^1 \end{cases} \tag{2}$$

and

$$A = \ldots \tag{3}$$

Solve (2) with $m^1(z,k) \to 1, m^2(z,k) \to 0$ as $|z| \to \infty$. Define the Scattering Transform:

$$\mathbf{s}(k) := \mathcal{S}q(k) = -\frac{i}{\pi} \int_{\mathbb{R}^2} e_k(z) \overline{q(z)} m^1(z,k) dz. \tag{4}$$

where $e_k(z) = e^{i(zk + \overline{zk})}$ and $dz = dx_1 dx_2$. Then

$$\frac{\partial}{\partial t} \mathbf{s}(t,k) = 2i(k^2 + \overline{k}^2) \mathbf{s}(t,k). \tag{5}$$

Using the Scattering Transform

The Inverse-Scattering Transform:

$$\mathcal{I}\mathbf{s}(z) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e_z(k)\overline{\mathbf{s}(k)}m^1(z,k)\,dk. \tag{6}$$

Can we solve the Cauchy problem for DSII with q_0 in L^2 as follows ?

$$\begin{cases} \mathbf{s}_0(k) & = \mathcal{S}q_0(k) \\ \mathbf{s}(t,k) & = e^{2i(k^2+\overline{k}^2)t}\mathbf{s}_0(k) \\ q(t,z) & = \mathcal{I}\big(\mathbf{s}(t,k)\big)(z). \end{cases} \tag{7}$$

$$\begin{array}{ccc} q_0(z) & \xrightarrow{\text{nonlin}} & q(t,z) \\ \Big\downarrow \mathcal{S} & & \mathcal{I} \Big\uparrow \\ \mathbf{s}_0(k) & \xrightarrow{\text{linear}} & \mathbf{s}(t,k). \end{array}$$

Nonlinear Plancherel Identity

Beals and Coifman (1998) proved that for q in Schwartz class $\mathbf{s}$ is in Schwartz class and the whole procedure is rigorous. Moreover they showed:

$$\int_{\mathbb{R}^2} |\mathbf{s}(k)|^2 dk = \int_{\mathbb{R}^2} |q(z)|^2 dz.$$

Open Problem: true for all q in L^2 ?

- R. Brown (2001) - q in L^2 with small norm
- P. Perry (2014) - q in weighted Sobolev space $H^{1,1}$
- K. Astala, D. Faraco and K. Rogers (2015) - q in weighted Sobolev space $H^{\varepsilon,\varepsilon}$, $\varepsilon > 0$
- R. Brown, K. Ott and P. Perry (2016) - $q \in H^{\alpha,\beta}$ iff $\mathbf{s} \in H^{\beta,\alpha}$, $\alpha, \beta > 0$

Plancherel Theorem

Theorem (N-Regev-Tataru)

The nonlinear scattering transform $\mathcal{S} : q \mapsto \mathbf{s}$ *is a* C^1 *diffeomorphism* $\mathcal{S} : L^2 \to L^2$, *satisfying:*

1. *The Plancherel Identity:* $\|\mathcal{S}q\|_{L^2} = \|q\|_{L^2}$
2. *The pointwise bound:* $|\mathcal{S}q(k)| \leq C(\|q\|_{L^2}) M\hat{q}(k)$ *for a.e.* k
3. *Locally uniform bi-Lipschitz continuity:*

$$\frac{1}{C}\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2} \leq \|q_1 - q_2\|_{L^2} \leq C\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2}$$

where

$$C = C(\|q_1\|_{L^2})C(\|q_2\|_{L^2}).$$

4. *Inversion Theorem:* $\mathcal{S}^{-1} = \mathcal{S}$.

A bit about the Proof

Making the substitution

$$m_\pm = m^1 \pm e_{-k} m^2,$$

we need to solve

$$\begin{cases} \frac{\partial}{\partial \bar{z}} m_\pm = \pm e_{-k} q \overline{m_\pm} \\ m_\pm \to 1 \text{ as } |z| \to \infty. \end{cases}$$

In integral form,

$$m_\pm - 1 = (\bar\partial \mp e_{-k} q \bar{\cdot})^{-1} \bar\partial^{-1}(e_{-k} q).$$

1. For $q \in L^2$, we need new bounds on $\bar\partial^{-1}(e_{-k} q)$ which allow us to capture the large k decay without assuming any smoothness on q.
2. We need bounds on $(\bar\partial \mp e_{-k} q \bar{\cdot})^{-1}$ which depend only on the L^2 norm of q.

New Estimate on Fractional Integrals

Lemma

For $q \in L^2(\mathbb{C})$,

$$\|\bar{\partial}^{-1}(e_{-k}q)\|_{L^4} \lesssim \|q\|_{L^2}^{\frac{1}{2}} \Big(M\hat{q}(k)\Big)^{\frac{1}{2}}.$$

M is the Hardy-Littlewood Maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

which yields a bounded operator on L^p for $1 < p \leq \infty$.

Theorem

For $0 < \alpha < n$, $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$

$$\left|(-\Delta)^{-\frac{\alpha}{2}} f(x)\right| \leq c_{n,\alpha} \Big(M\hat{f}(0)\Big)^{\frac{\alpha}{n}} \Big(Mf(x)\Big)^{1-\frac{\alpha}{n}}$$

Sketch of Proof - Fractional Integrals

Proof.

Using Littlewood-Paley decomposition,

$$(-\Delta)^{-\frac{\alpha}{2}} f(x) = \frac{1}{(2\pi)^n} \sum_{j=-\infty}^{j_0} \int_{\mathbb{R}^n} \psi_j(\xi) \frac{e^{ix\cdot\xi}}{|\xi|^\alpha} \hat{f}(\xi) d\xi + \sum_{j_0+1}^{\infty} \dots$$

with $\psi_j(\xi) = \psi(\xi/2^j)$ supported in $2^{j-1} < |\xi| < 2^{j+1}$. For $j \leq j_0$ use

$$\int_{|\xi|<r} |\hat{f}(\xi)| d\xi \leq c_n r^n M\hat{f}(0)$$

...

$$\left|(-\Delta)^{-\frac{\alpha}{2}} f(x)\right| \lesssim 2^{j_0(n-\alpha)} M\hat{f}(0) + 2^{-j_0\alpha} Mf(x)$$

optimize over j_0. □

Key Theorem - bounds in terms of $\|q\|_{L^2}$

Theorem

Let $q \in L^2$. Then for each $f \in \dot{H}^{-\frac{1}{2}}$ there exists a unique solution $u \in \dot{H}^{\frac{1}{2}}$ of

$$L_q u := \bar{\partial} u + q\bar{u} = f \tag{8}$$

with

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2})\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{9}$$

In particular, for $f \in L^{\frac{4}{3}}$ the same holds, with $\|u\|_{L^4} \leq C(\|q\|_{L^2})\|f\|_{L^{\frac{4}{3}}}$.

Idea of the proof: use Induction on Energy and Profile Decompositions to study the static problem.

Construction of the Jost Solutions for $q \in L^2$

As a result of the new estimates on fractional integrals and the Key Theorem, we can now establish

Theorem (Jost Solutions)

Suppose $q \in L^2$, then for almost every k there exist unique Jost solutions $m_\pm(z,k)$ with $m_\pm(\cdot,k) - 1 \in L^4$ and moreover

$$\|m(\cdot,k)_\pm - 1\|_{L^4} \leq C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}$$

$$\|m_\pm - 1\|_{L^4_z L^4_k} \leq C(\|q\|_{L^2}).$$

$$\|\bar{\partial} m^1(\cdot,k)\|_{L^{\frac{4}{3}}} \leq C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}.$$

Scattering Transform as a ΨDO

Recall

$$\mathbf{s}(k) = \hat{\overline{q}}(k) - \frac{i}{\pi} \int_{\mathbb{R}^2} e_k(z)\overline{q(z)}(m^1(z,k) - 1)dz.$$

Replace $\overline{q}$ by the Fourier transform of some function in L^2. Then the above becomes a pseudo-differential operator with symbol $m^1 - 1$. We'd like to prove it is a bounded operator on L^2.

Theorem

Let $0 \leq \alpha < n$. Suppose $a(x,\xi)$ satisfies

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |a(x,\xi)|^{\frac{2n}{n-\alpha}} dx d\xi < \infty \qquad \text{and} \qquad \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi)\|_{L_\xi^{\frac{2n}{n+\alpha}}} \in L_x^{\frac{2n}{n-\alpha}}.$$

Then the pseudo-differential operator

$$a(x,D)f(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)d\xi \tag{10}$$

is bounded on L^2. Moreover, we have the pointwise bound

$$|a(x,D)f(x)| \leq c_{\alpha,n}(Mf(x))^{\alpha/n}\|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\cdot)\|_{L^{\frac{2n}{n+\alpha}}}\|f\|_{L^2}^{1-\frac{\alpha}{n}} \tag{11}$$

for a.e. x.

This completes the sketch of the proof of the Plancherel Theorem.

GWP for Defocusing DSII on L^2

Theorem

Given $q_0 \in L^2$, there exists a unique solution to the Cauchy Problem for defocusing DSII such that:

1. *Regularity:*
$$q(t,z) \in C(\mathbb{R}, L^2_z(\mathbb{C})) \cap L^4_{t,z}(\mathbb{R} \times \mathbb{C}).$$
2. *Uniform bounds:* $\|q(t,\cdot)\|_{L^2} = \|q_0\|_{L^2}$ *for all* $t \in \mathbb{R}$ *and*
$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |q(t,z)|^4 dzdt \leq C(\|q_0\|_{L^2}).$$
3. *Stability: if* $q_1(t,\cdot)$ *and* $q_2(t,\cdot)$ *are two solutions corresponding to initial data* $q_1(0,\cdot)$ *and* $q_2(0,\cdot)$ *with* $\|q_j(0,\cdot)\|_{L^2} \leq R$ *then*
$$\|q_1(t,\cdot) - q_2(t,\cdot)\|_{L^2} \leq C(R)\|q_1(0,\cdot) - q_2(0,\cdot)\|_{L^2} \qquad \text{for all } t \in \mathbb{R}.$$

Proof that $q(t, z) \in L^4_{t,z}(\mathbb{R} \times \mathbb{C})$

$$\mathbf{s}(t, k) = e^{2i(k^2+\overline{k}^2)t}\mathbf{s}_0(k)$$

$$\begin{aligned} |q(t, z)| &= |\mathcal{S}^{-1}(\mathbf{s}(t, \cdot))(z)| \\ &\leq C(\|q_0\|_{L^2}) M\check{\mathbf{s}}(t, z) \end{aligned}$$

where

$$\check{\mathbf{s}}(t, z) = \int e_z(k) e^{2i(k^2+\overline{k}^2)t}\mathbf{s}_0(k)dk := U(t)(\check{\mathbf{s}}_0)(z)$$

is linear flow starting from $\check{\mathbf{s}}_0$ for which we have the Strichartz estimate

$$\|\check{\mathbf{s}}\|_{L^4_{t,z}} \lesssim \|\check{\mathbf{s}}_0\|_{L^2} = \|\mathbf{s}_0\|_{L^2} = \|q_0\|_{L^2}.$$

Time-domain Scattering

The Scattering Transform also yields the large time behaviour of the solutions to the DSII equation. Recall the definition of the wave operators, in the sense of nonlinear scattering theory.

Definition

Let $q_0 \in L^2(\mathbb{R}^2)$ and let $q(t,z)$ be the solution to the Cauchy problem for defocusing DSII. Define $W_+ q_0 = q_+$ if there exists a unique $q_+ \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t\to\infty} \|q(t,\cdot) - U(t)q_+\|_{L^2(\mathbb{R}^2)} = 0.$$

Similarly $W_- q_0 = q_-$ if

$$\lim_{t\to-\infty} \|q(t,\cdot) - U(t)q_-\|_{L^2(\mathbb{R}^2)} = 0.$$

Wave operators and asymptotic completeness for defocusing DSII

Theorem

a) The Wave operators $W_\pm$ for the defocusing DSII equation are well defined on every $q_0 \in L^2(\mathbb{R}^2)$ and

$$W_\pm q_0 = \check{\mathcal{S}} q_0.$$

b) The Wave operators $W_\pm$ are surjective, in fact norm-preserving diffeomorphisms of L^2.

Perry (2014) established the same large time asymptotic behaviour in the L^∞ norm, for initial data in $H^{1,1} \cap L^1$.

An interesting consequence of the Theorem is that the temporal scattering operator $W_+(W_-)^{-1}$ for the defocusing DSII equation (i.e. the operator which sends q_- to q_+) is equal to the identity.

The Calderón Inverse Conductivity Problem

Let Ω be a simply connected domain in $\mathbb{R}^2 \simeq \mathbb{C}$

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} = g. \end{cases} \tag{12}$$

The Dirichlet-to-Neumann map is defined as

$$\Lambda_\sigma f := \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

A.P. Calderón (1980) posed the problem: does Λ_σ uniquely determine σ?

- N. (1996) - Unique reconstruction for $\sigma \in W^{2,p}(\Omega)$ for some $p > 1$
- R. Brown. G. Uhlman (1997) - $\sigma \in W^{1,p}(\Omega)$, for some $p > 2$.
- K. Astala, L. Päivärinta (2006) - $\sigma \in L^\infty$
- K. Astala, M. Lassas, L. Päivärinta (2016) - Larger class of conductivities which includes some unbounded ones.
- C.Carstea J.-N. Wang $\log \sigma \in L^2(\Omega)$ with small norm (2018)

The Calderón Inverse Conductivity Problem

Theorem

Suppose $\sigma > 0$ is such that $\nabla \log \sigma \in L^2(\Omega)$ and $\sigma = 1$ on $\partial\Omega$, then we can reconstruct σ from knowledge of Λ_σ.

Outline of the proof:

Let $v = \sigma^{\frac{1}{2}} \partial u$ then for u real valued, $\bar{\partial} v = q\overline{v}$ where $q = -\frac{1}{2}\partial \log \sigma \in L^2$.

$$\begin{aligned}
\mathbf{s}(k) &= \frac{1}{2\pi i} \int_{\mathbb{R}^2} e_k(z)\overline{q(z)}\Big(m_+(\cdot, k) + m_-(\cdot, k)\Big) \\
&= \frac{1}{2\pi i} \int_{\Omega} \partial\Big(\overline{m_+}(\cdot, k) - \overline{m_-}(\cdot, k)\Big) \\
&= \frac{1}{4\pi i} \int_{\partial\Omega} \overline{\nu}\Big(\overline{m_+}(\cdot, k) - \overline{m_-}(\cdot, k)\Big)
\end{aligned}$$

Proof consists in showing that Λ_σ determines the traces of $m_\pm(\cdot, k)$ on $\partial\Omega$.

Thank You!