Metastability for Interacting Particle Systems

II. Glauber dynamics on random graphs

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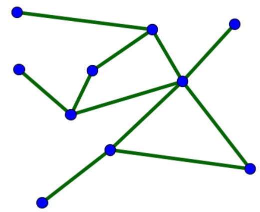
Online Open Probability School, June 25, 2020,
supported by PIMS, CRM, SMS, BIRS, MSRI.

§ GLAUBER DYNAMICS ON GRAPHS



Let $G = (V, E)$ be a connected graph. Ising spins are attached to the vertices V and interact with each other along the edges E.

1. The energy associated with the configuration $\sigma = (\sigma_i)_{i \in V} \in \Omega = \{-1, +1\}^V$ is given by the Hamiltonian

$$\boxed{H(\sigma) = -J \sum_{(i,j) \in E} \sigma_i \sigma_j - h \sum_{i \in V} \sigma_i}$$

where $J > 0$ is the ferromagnetic interaction strength and $h > 0$ is the external magnetic field.

2. Spins flip according to Glauber dynamics

$$\boxed{\forall\,\sigma \in \Omega\ \forall\, j \in V:\quad \sigma \to \sigma^j \text{ at rate } \mathrm{e}^{-\beta[H(\sigma^j)-H(\sigma)]_+}}$$

where σ^j is the configuration obtained from σ by flipping the spin at vertex j, and $\beta > 0$ is the inverse temperature.

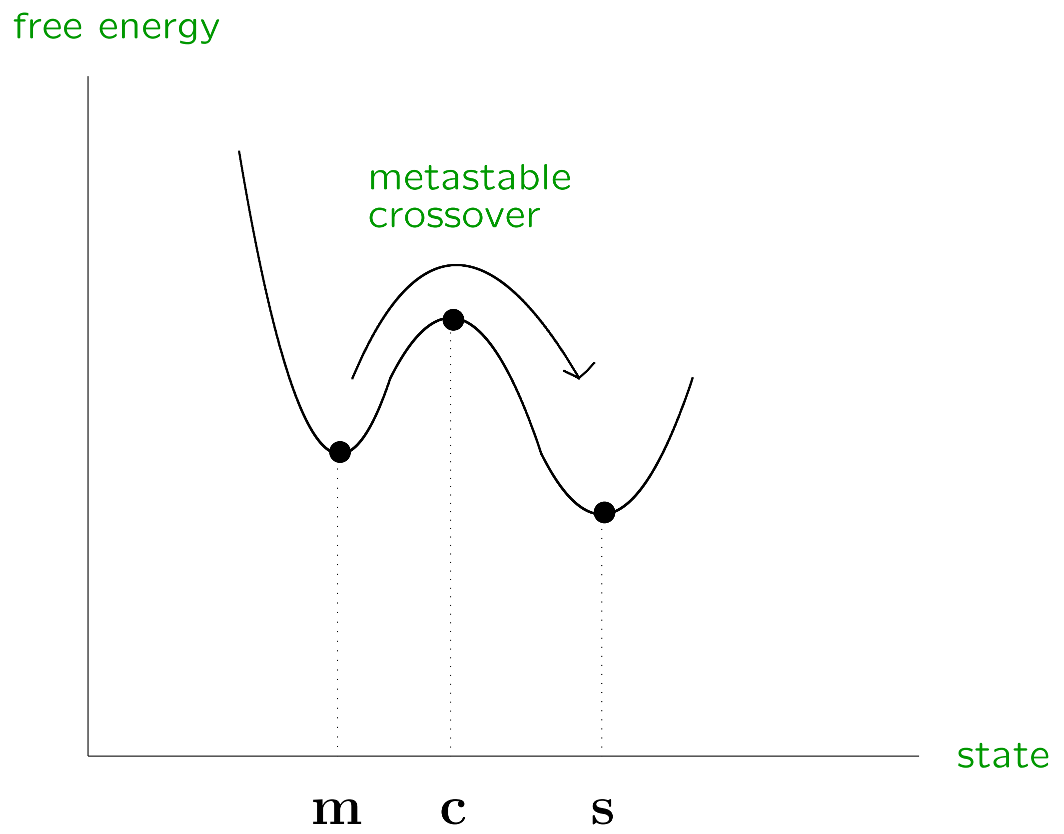
3. The Gibbs measure

$$\mu(\sigma) = \frac{1}{\Xi}\,\mathrm{e}^{-\beta H(\sigma)}, \qquad \sigma \in \Omega,$$

is the reversible equilibrium of this dynamics.

4. Three sets of configurations play a central role:

$$\mathbf{m} = \text{metastable state}$$
$$\mathbf{c} = \text{crossover state}$$
$$\mathbf{s} = \text{stable state}.$$



Caricature of the free energy landscape

– energy and entropy –

THEOREM: Bovier, Eckhoff, Gayrard, Klein 2001

Let $P_{\mathbf{m}}$ *denote the probability distribution on path space of the Glauber dynamics starting at* $\mathbf{m}$*. Let* $\tau_{\mathbf{s}}$ *denote the first hitting time of* $\mathbf{s}$*. Then*

$$\boxed{E_{\mathbf{m}}(\tau_{\mathbf{s}}) = [1 + o(1)]\,\frac{\mu(\mathbf{m})}{\mathrm{cap}(\mathbf{m},\mathbf{s})}}$$

with $\mathrm{cap}(\mathbf{m},\mathbf{s})$ *the capacity of the pair* $(\mathbf{m},\mathbf{s})$*.*

Here, $o(1)$ refers to a parameter regime where the system is metastable, e.g. for low temperature or large volume.

Recall the Dirichlet Principle

$$\mathrm{cap}(\mathbf{m},\mathbf{s}) = \inf_{\phi\in\Phi_{\mathbf{m},\mathbf{s}}} \mathcal{E}(\phi,\phi)$$

with

$$\mathcal{E}(\phi,\phi) = \frac{1}{\Xi} \sum_{\substack{\sigma,\sigma'\in\Omega \\ \sigma\sim\sigma'}} e^{-\beta[H(\sigma)\vee H(\sigma')]} [\phi(\sigma')-\phi(\sigma)]^2,$$

$$\Phi_{\mathbf{m},\mathbf{s}} = \{\phi\colon\ \Omega\to[0,1]\colon\ \phi(\mathbf{m})=1,\ \phi(\mathbf{s})=0\}$$

and the Thomson Principle

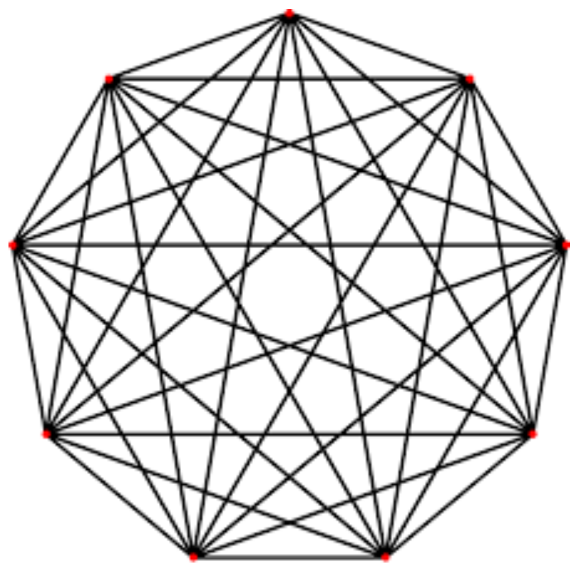
$$\text{cap}(\mathbf{m},\mathbf{s}) = \sup_{u \in \mathcal{U}_{\mathbf{m},\mathbf{s}}} \frac{1}{\mathcal{D}(u,u)}$$

with

$$\mathcal{D}(u,u) = \frac{1}{\Xi} \sum_{\substack{\sigma,\sigma' \in \Omega \\ \sigma \sim \sigma'}} \mathrm{e}^{\beta[H(\sigma) \vee H(\sigma')]}\, u(\sigma,\sigma')^2,$$

$$\mathcal{U}_{\mathbf{m},\mathbf{s}} = \{u\colon\ \Omega \times \Omega \to [0,1]\colon\ u \text{ is a unit flow}\}.$$

§ COMPLETE GRAPH



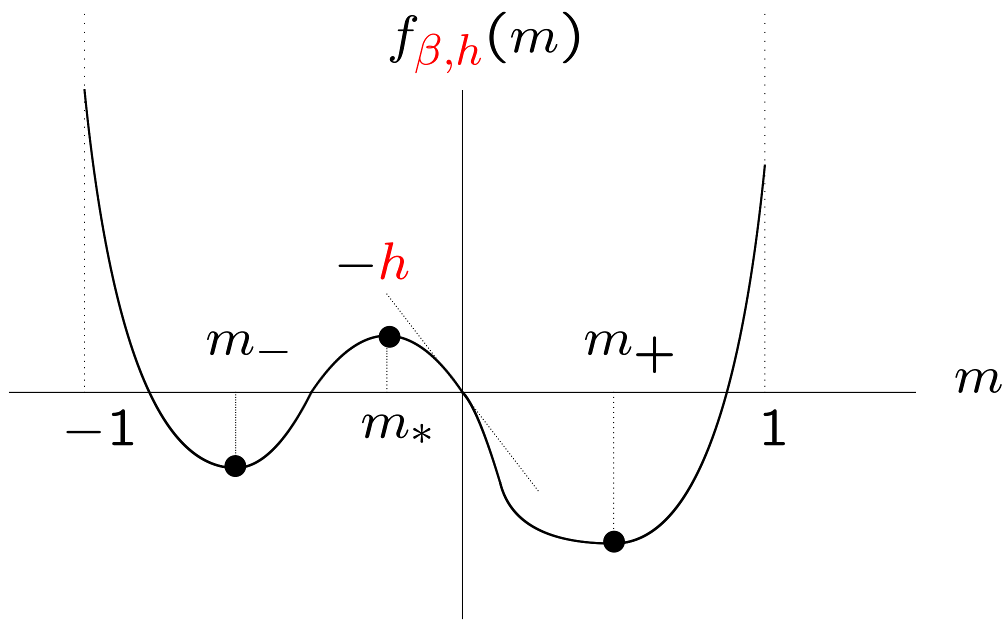
Complete graph: Curie-Weiss

Put $J = 1/N$. Then in the limit as $N \to \infty$, the free energy per vertex when the magnetization is m equals

$$f_{\beta,h}(m) = -\frac{1}{2}m^2 - hm + \beta^{-1}I(m)$$

with

$$I(m) = \frac{1}{2}(1+m)\log(1+m) + \frac{1}{2}(1-m)\log(1-m).$$



EXERCISE!

THEOREM: Bovier, Eckhoff, Gayrard, Klein 2001

On the complete graph with N *vertices, for* $J = 1/N$, $\beta > 1$ *and* $h \in (0, \chi(\beta))$,

$$E^{\mathsf{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}) = K\, e^{N\Gamma}[1 + o(1)], \qquad N \to \infty,$$

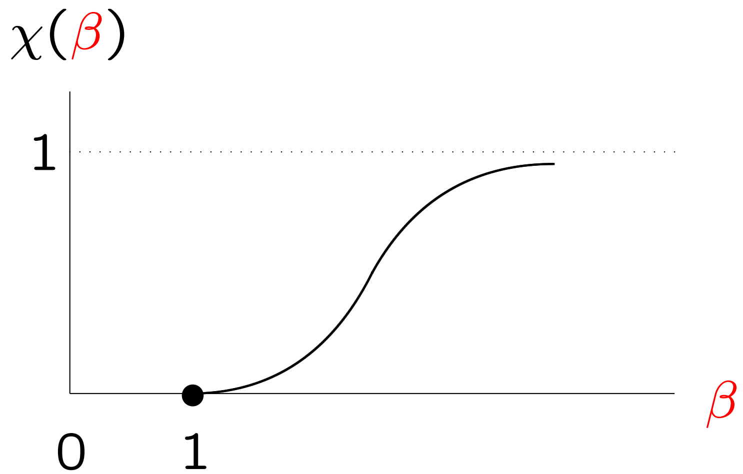
where $\mathbf{m}_N^-, \mathbf{m}_N^+$ *are the sets of configurations for which the magnetization tends to* m_-, m_+,

$$\Gamma = \beta\, [f_{\beta,h}(m_*) - f_{\beta,h}(m_-)]$$

$$K = \pi\beta^{-1} \sqrt{\frac{1-m_*}{1+m_*}\, \frac{1}{1-m_-^2}\, \frac{1}{[-f''_{\beta,h}(m_*)]\, f''_{\beta,h}(m_-)}}$$

and

$$\chi(\beta) = \sqrt{1 - \tfrac{1}{\beta}} - \tfrac{1}{2\beta} \log\left[\beta\left(1 + \sqrt{1 - \tfrac{1}{\beta}}\right)^2\right].$$



The conditions on β and h are needed to ensure that the free energy $m \mapsto f_{\beta,h}(m)$ has a double-well structure.

NOTE: The asymptotics for the crossover time is uniform in the starting configuration drawn from the set $\mathbf{m}_N$.

The proof uses a lumping technique typical for mean-field models: for finite N the magnetization performs a random walk on the $\frac{2}{N}$-grid in $[-1,1]$ in a potential close to $f_{\beta,h}$.

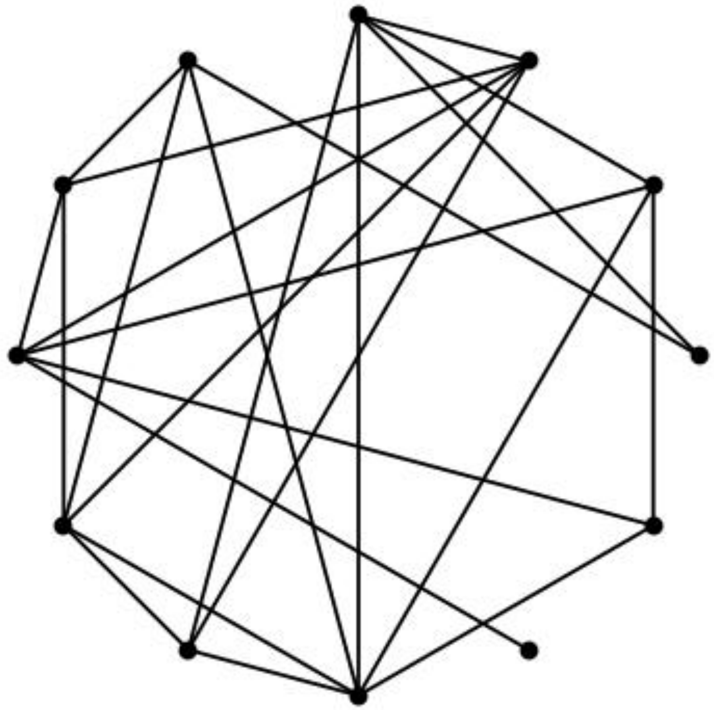
In the limit as $N \to \infty$, the system behaves like

$$\boxed{\text{Brownian motion in a double-well potential}}$$

analysed by Kramers 1940.

EXERCISE!

§ ERDŐS-RÉNYI RANDOM GRAPH



Erdős-Rényi random graph: edge percolation

Take the complete graph with N vertices and retain edges with probability $p \in (0, 1)$.

Two parallel projects:

Frank den Hollander

Oliver Jovanovksi

Anton Bovier

Saeda Marello

Elena Pulvirenti

THEOREM: den Hollander, Jovanovski 2019

On the Erdős-Rényi random graph with N vertices, for $J = 1/pN$, $\beta > 1$ and $h \in (0, \chi(\beta))$,

$$E^{\mathrm{ER}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}) = N^{E_N}\, E^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}), \qquad N \to \infty,$$

where E_N is a random exponent that satisfies

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{ER}_{\mathrm{N}}(\mathrm{p})}\Big(|E_N| \leq \tfrac{11}{6}\tfrac{\beta}{p}(m^* - m_-)\Big) = 1,$$

where $\mathbb{P}_{\mathrm{ER}_{\mathrm{N}}(\mathrm{p})}$ is the law of the random graph.

Apart from a polynomial error term, the crossover time is the same as on the complete graph.

The asymptotic estimate of the crossover time is again uniform in the starting configuration drawn from the set $\mathbf{m}_N$.

Note that J needs to be scaled up by a factor $1/p$ in order to allow for a comparison with the Curie-Weiss model: in the Erdős-Rényi model every spin interacts with $\sim pN$ spins rather than N spins.

The latter observation also explains why we do not consider the non-dense Erdős-Rényi random graph with $p = p_N \downarrow 0$ as $N \to \infty$.

On the complete graph the prefactor is constant in N and is computable. On the Erdős-Rényi random graph it is more involved.

§ REFINEMENT OF THE PREFACTOR

THEOREM: Bovier, Marello, Pulvirenti 2019

For $\beta > 1$, $h > 0$ *small enough and* $s > 0$*,*

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{ER}_n(p)}\left(C_1 \mathrm{e}^{-s} \le \frac{E^{\mathrm{ER}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+})}{E^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+})} \le C_2 \mathrm{e}^{s} \right) \ge 1 - k_1 \mathrm{e}^{-k_2 s^2},$$

where $k_1, k_2 > 0$ *are absolute constants, and* $C_1 = C_1(p, \beta)$ *and* $C_2 = C_2(p, \beta, h)$*.*

This theorem shows that the prefactor is a tight random variable, and hence constitutes a considerable sharpening of the previous theorem.

The sharp control on the prefactor comes at a price:

- The magnetic field is taken small enough.
- The dynamics starts according to the last-exit biased distribution for the transition from $\mathbf{m}_N$ to $\mathbf{s}_N$, rather than from an arbitrary configuration in $\mathbf{m}_N$.

The estimate is stated and proved for discrete-time rather than continuous-time dynamics: at each unit of time only one spin is allowed to flip. This means that time runs slower by a factor N.

Proofs rely on elaborate techniques:

isoperimetric inequalities
concentration estimates
capacity estimates
coupling techniques
coarse-graining techniques
...

These all exploit the fact that, in the dense regime, the Erdős-Rényi random graph is highly homogeneous.

§ RANDOM MAGNETIC FIELD

An interesting model is where the randomness sits on the vertices rather than on the edges, namely,

$$H(\sigma) = -\frac{1}{N} \sum_{1 \leq i,j \leq N} \sigma_i \sigma_j - \sum_{1 \leq i \leq N} h_i \sigma_i,$$

where h_i, $1 \leq i \leq N$, are i.i.d. random variables drawn from a common probability distribution ν on $\mathbb{R}$.

Bovier, Eckhoff, Gayrard, Klein 2001 ν discrete

Bianchi, Bovier, Ioffe 2009 + 2012 ν continuous

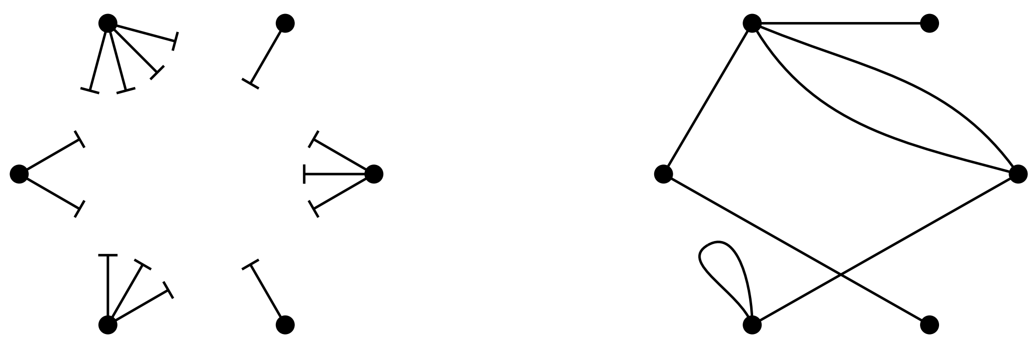
The prefactor turns out to be constant in N and to be a somewhat involved function of ν.

Our model is harder because the interaction runs along the set of edges, which has an intricate spatial structure.

Lumping technniques cannot be used: in the above papers the magnetization is monitored on the level sets of the magnetic field.

§ RELATED WORK

What can be said in the sparse regime after a proper scaling of the interaction strength?



Rough estimates for the average metastable crossover time are known for the configuration model (a random graph with prescribed degrees) when N, J, h are fixed and $\beta \to \infty$.

Dommers 2017

Dommers, den Hollander, Jovanovski, Nardi 2017

Jovanovski 2017

Metastability for Kawasaki dynamics on random graphs in the sparse regime also poses serious challenges.

Some progress has been made for bi-partite graphs:

den Hollander, Nardi, Taati 2018

Borst, den Hollander, Nardi, Taati 2020

Borst, den Hollander, Nardi, Sfragara 2020

TAKE-HOME MESSAGE

Prefactors of average metastable crossover times are delicate objects for random graphs, because they depend in an intricate manner on the underlying geometry.

Very little is known so far

and much remains to be done!

PAPERS:

(1) F. den Hollander, O. Jovanovski,
Glauber dynamics on the Erdős-Rényi random graph,
preprint 2019 [arXiv:1912.10591].

(2) A. Bovier, S. Marello, E. Pulvirenti,
Metastability for the dilute Curie-Weiss model with Glauber dynamics,
preprint 2019 [arXiv:1912.10699].