High-order compact finite difference schemes for option pricing

Bertram Düring

Department of Mathematics

UNIVERSITY OF SUSSEX

Joint work with

M. Fournié, C. Heuer, J. Miles, A. Pitkin, A. Rigal

Stochastic volatility model: Heston (1993)

Let $W = (W^{(1)}, W^{(2)})$ denote a two-dimensional Brownian motion with correlation $dW^{(1)}(t)dW^{(2)}(t) = \rho dt$

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Underlying asset $S(t)$ follows

$$dS(t) = \bar{\mu}S(t)\,dt + \sqrt{\sigma(t)}S(t)\,dW^{(1)}(t),$$
$$d\sigma(t) = \kappa^*(\theta^* - \sigma(t))\,dt + v\sqrt{\sigma(t)}\,dW^{(2)}(t),$$

for $0 < t \leq T$ with $S(0), \sigma(0) > 0$.

$\bar{\mu}$: drift
κ^*: mean reversion speed
v: volatility of volatility
θ^*: long-run mean of σ

Heston PDE

Option price $V = V(S, \sigma, t)$ solves

$$V_t + \frac{1}{2}S^2\sigma V_{SS} + \rho v \sigma S V_{S\sigma} + \frac{1}{2}v^2\sigma V_{\sigma\sigma} + rSV_S \\ + \left[\kappa^*(\theta^* - \sigma) - \lambda\sigma\right]V_\sigma - rV = 0,$$

for $S, \sigma > 0$, $0 \leq t < T$ and subject to, e.g., for the put option

$$V(S, \sigma, T) = \max(K - S, 0)$$

and suitable boundary conditions

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$\rightarrow$ for *constant* parameters there exists a closed form solution
$\rightarrow$ in general has to be solved numerically

Literature (incomplete)

Finite difference literature:

- Ikonen/Toivanen (2007): compare different efficient, 2nd order methods for solving American option pricing problem
- in't Hout/Foulon (2007): adapt different, 2nd order ADI schemes to include mixed spatial derivative term
- Tangman *et. al* (2008): compact scheme for 1d case, remark on 2D case, final scheme is low order

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Other approaches: finite element-finite volume (Zvan *et. al*, 1998), multigrid (Clarke/Parrott, 1999), sparse wavelet (Hilber *et. al*, 2005), spectral methods (Zhu/Kopriva, 2010), FFT-based (Osterlee *et. al*, 2012), RBF-FD (v. Sydow *et. al*, 2015), ...

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→ Aim: derive high-order compact finite difference scheme

High-order schemes

Higher-order approximation (e.g. fourth-order in spatial discretisation parameter) can be obtained by increasing the width of the computational stencil, e.g.

$$(u_{xx})_i \approx \frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12\Delta x^2}$$

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However:

→ leads to increased bandwidth of the discretisation matrices
→ complicates formulations of boundary conditions
→ such approaches sometimes suffer from restrictive stability conditions and spurious numerical oscillations

High-order compact schemes

These problems do not arise when using a compact stencil, e.g. in 2D: use nine-point computational stencil involving the eight nearest neighboring points of the reference grid point (i, j):

$$\begin{pmatrix} u_{i-1,j+1} & u_{i,j+1} & u_{i+1,j+1} \\ u_{i-1,j} & u_{i,j} & u_{i+1,j} \\ u_{i-1,j-1} & u_{i,j-1} & u_{i+1,j-1} \end{pmatrix}$$

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$\rightarrow$ how to obtain high-order consistency?

Idea: operate on the differential equation as auxiliary relation to obtain finite difference approximations for high-order derivatives in the truncation error of a lower-order approximation

High-order compact schemes: literature

High-order compact schemes for

- elliptic problems: Collatz ('74), (Gupta et al. ('84,'85), Spotz & Carey ('96)
- parabolic problems (isotropic): Spotz & Carey ('01), Karaa & Zhang ('02)
- fully nonlinear parabolic PDEs: B.D., Fournié & Jüngel ('03,'04)
- anisotropic, elliptic PDE, constant coefficients: Fournié & Karaa ('06)

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for $S, \sigma > 0$, $0 \le t < T$ and subject to, e.g., for the put option

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Parameters and boundary conditions

Introducing modified parameters

$$\kappa = \kappa^* + \lambda, \quad \theta = \kappa^* \theta^* / (\kappa^* + \lambda)$$

allows to study the problem with one parameter less.

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Boundary conditions for the put option are

$$V(0, \sigma, t) = Ke^{-r(T-t)}, \quad T > t \geq 0, \ \sigma > 0$$
$$V(S, \sigma, t) \to 0, \quad T > t \geq 0, \ \sigma > 0, \text{ as } S \to \infty$$
$$V_\sigma(S, \sigma, t) \to 0, \quad T > t \geq 0, \ S > 0, \text{ as } \sigma \to \infty$$
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Transformation of the equation

Using

$$x = \ln(S/K), \quad y = \sigma/v, \quad \tilde{t} = T - t, \quad u = \exp(r\tilde{t})V/K,$$

we obtain

$$u_t - \frac{1}{2}vy(u_{xx} + u_{yy}) - \rho v y u_{xy} + \left(\frac{1}{2}vy - r\right)u_x - \kappa\frac{\theta - vy}{v}u_y = 0,$$

to be solved on $\mathbb{R} \times \mathbb{R}^+$ with initial and boundary conditions.

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B.D. and M. Fournié.
High-order compact finite difference scheme for option pricing in stochastic volatility models.
J. Comput. Appl. Math. **236**(17), 2012. (arXiv:1404.5140)

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Use nine-point computational stencil involving the eight nearest neighboring points of the reference grid point (i, j):

$$\begin{pmatrix} u_{i-1,j+1} = u_6 & u_{i,j+1} = u_2 & u_{i+1,j+1} = u_5 \\ u_{i-1,j} = u_3 & u_{i,j} = u_0 & u_{i+1,j} = u_1 \\ u_{i-1,j-1} = u_7 & u_{i,j-1} = u_4 & u_{i+1,j-1} = u_8 \end{pmatrix}.$$

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$\rightarrow$ consider first the elliptic problem with right-hand side f

Derivation of the high-order compact scheme

Introduce uniform grid with mesh spacing h in both the x- and y-direction, standard central difference approximation is

$$
\begin{aligned}
&-\frac{1}{2}vy_j\big(\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j}\big) - \rho v y_j \delta_x \delta_y u_{i,j} \\
&\qquad + \left(\frac{1}{2}vy_j - r\right)\delta_x u_{i,j} - \kappa\frac{\theta - vy_j}{v}\delta_y u_{i,j} - \tau_{i,j} = f_{i,j},
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where δ_x, δ_x^2 (δ_y, δ_y^2, respectively) denote the first and second order central difference approximations

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Truncation error is given by

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\begin{aligned}
\tau_{i,j} = {} & \frac{1}{24}vyh^2(u_{xxxx} + u_{yyyy}) + \frac{1}{6}\rho v y h^2 (u_{xyyy} + u_{xxxy}) \\
& + \frac{1}{12}(2\,r - vy)h^2 u_{xxx} + \frac{1}{6}\frac{\kappa(\theta - vy)}{v}h^2 u_{yyy} + \mathcal{O}(h^4)
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$\rightarrow$ seek second-order approximations to the derivatives

Derivation of the high-order compact scheme

Substituting these expressions the truncation error yields a new expression for the error term $\tau_{i,j}$ that consists only of terms which are either

- terms of order $\mathcal{O}(h^4)$, or
- terms of order $\mathcal{O}(h^2)$ multiplied by derivatives of u which can be approximated up to $\mathcal{O}(h^2)$ within the nine-point compact stencil

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$\rightarrow$ inserting all into the central difference approximation of the equation yields a $\mathcal{O}(h^4)$ approximation to the elliptic Heston PDE

$$\sum_{l=0}^{8} \alpha_l u_l = \sum_{l=0}^{8} \gamma_l f_l,$$

with given coefficients α_l and γ_l

Time integration

Considering the time derivative in place of $f(x, y)$
$\rightarrow$ any time integrator can be implemented
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For example, differencing at $t_\mu = (1 - \mu)t^n + \mu t^{n+1}$, where $0 \leq \mu \leq 1$ yields a class of integrators that include the forward Euler ($\mu = 0$), Crank-Nicolson ($\mu = 1/2$) and backward Euler ($\mu = 1$) schemes.

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Resulting fully discrete difference scheme for node (i, j)

$$\sum_{l=0}^{8} \mu \alpha_l u_l^{n+1} + (1-\mu)\alpha_l u_l^n = \sum_{l=0}^{8} \gamma_l \delta_t^+ u_l^n,$$

with $\delta_t^+ u^n = \frac{u^{n+1} - u^n}{k}$

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with $\delta_t^+ u^n = \frac{u^{n+1} - u^n}{k}$
$\rightarrow$ for $\mu = 1/2$ the scheme is of order two in time and of order four in space

Initial condition

The initial condition is given by the transformed payoff function of the put option,

$$u(x, \mathbf{y}, 0) = \max(1 - \exp(x), 0), \quad x \in \mathbb{R}, \;\; \mathbf{y} > 0.$$

- Kreiss (1970) states that we cannot achieve fourth order convergence if the initial condition is not sufficiently smooth.
- Smoothing operators are defined in the Fourier space, we apply smoothing operator to the initial condition

$$\tilde{u}_0(x_1, x_2) = \frac{1}{h^2} \int_{-3h}^{3h} \int_{-3h}^{3h} \phi_4\left(\frac{x}{h}\right) \phi_4\left(\frac{y}{h}\right) u_0(x_1 - x, x_2 - y) dx dy.$$

Final scheme

Final scheme can be written as $\sum_{l=0}^{8} \beta_l u_l^{n+1} = \sum_{l=0}^{8} \zeta_l u_l^n$ with

$$
\begin{aligned}
\beta_0 =& (((2{y_j}^2 - 8)v^4 + ((-8\kappa - 8r)y_j - 8\rho r)v^3 + (8\kappa^2 {y_j}^2 + 8r^2)v^2 \\
&- 16\kappa^2\theta v y_j + 8\kappa^2\theta^2)\mu k + 16v^3 y_j)h^2 + (-16\rho^2 + 40){y_j}^2 v^4 \mu k \\
\beta_{1,3} =& \pm ((\kappa\theta v^2 - v^4 - \kappa y_j v^3)\mu k - (y_j + 2\rho)v^3 + 2v^2 r)h^3 + (((-{y_j}^2 + 2)v^4 \\
&+ ((4r + 2\kappa)y_j + 4\rho r)v^3 - (2\kappa\theta + 4r^2)v^2)\mu k + 2v^3 y_j)h^2 \\
&\pm (4v^4 {y_j}^2 + (-8{y_j}^2\kappa\rho - 8y_j r)v^3 + 8y_j\kappa\theta\rho v^2)\mu k h + (8\rho^2 - 8){y_j}^2 v^4 \mu k, \\
\beta_{2,4} =& \pm ((2\kappa^2\theta v - 2\kappa^2 v^2 y_j - 2v^3\kappa)\mu k - 2v^2 y_j\kappa + 2v\kappa\theta - 2v^3)h^3 + ((2v^4 \\
&+ 2\kappa y_j v^3 + (-4\kappa^2 {y_j}^2 + 2\kappa\theta)v^2 + 8\kappa^2\theta v y_j - 4\kappa^2\theta^2)\mu k + 2v^3 y_j)h^2 \\
&\pm ((8{y_j}^2\kappa + 8y_j\rho r)v^3 - 4v^4 {y_j}^2\rho - 8v^2 y_j\kappa\theta)\mu k h + (8\rho^2 - 8){y_j}^2 v^4 \mu k, \\
\beta_{5,7} =& ((v^4\rho + (-y^2\kappa + \kappa y_j\rho + r)v^3 + (\theta + 2r)\kappa y_j v^2 - 2r\kappa\theta v)\mu k \\
&+ v^3\rho y_j)h^2 \pm ((2\rho + 1){y_j}^2 v^4 + ((2 + 4\rho)\kappa {y_j}^2 + (-4\rho r - 2r)y_j)v^3 \\
&+ (-2\theta - 4\theta\rho)\kappa y_j v^2)\mu k h + (-2 - 4\rho^2 - 6\rho){y_j}^2 v^4 \mu k, \\
\beta_{6,8} =& ((-v^4\rho + ({y_j}^2\kappa - \kappa y_j\rho - r)v^3 + (-\theta - 2r)\kappa y_j v^2 + 2r\kappa\theta v)\mu k \\
&- v^3\rho y_j)h^2 \pm ((2\rho - 1){y_j}^2 v^4 + ((2 - 4\rho)\kappa {y_j}^2 + (2r - 4\rho r)y_j)v^3 \\
&+ (4\theta\rho - 2\theta)\kappa y_j v^2)\mu k h + (-4\rho^2 + 6\rho - 2){y_j}^2 v^4 \mu k,
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$$
\begin{aligned}
\zeta_0 =& 16v^3 y_j h^2 + (1-\mu)k(((8-2{y_j}^2)v^4 + ((8\kappa + 8r)y_j + 8\rho r)v^3 \\
&+ (-8r^2 - 8\kappa^2 {y_j}^2)v^2 + 16\kappa^2\theta v y_j - 8\kappa^2\theta^2)h^2 + (-40 + 16\rho^2){y_j}^2 v^4), \\
\zeta_{1,3} =& \pm(2r - (y_j + 2\rho)v)v^2 h^3 + 2v^3 y_j h^2 + (1-\mu)k(\pm(v\kappa y_j + v^2 - \kappa\theta)v^2 h^3 \\
&+ (v^2 {y_j}^2 - (4r + 2\kappa)v y_j + 4r^2 + 2\kappa\theta - 2v^2 - 4\rho v r)v^2 h^2 \\
&\pm((-4v + 8\kappa\rho)v^3 {y_j}^2 + (-8\kappa\theta\rho + 8vr)v^2 y_j)h + (8v^2 - 8v^2\rho^2)v^2 {y_j}^2), \\
\zeta_{2,4} =& \pm(2v\kappa\theta - 2v^2 y_j \kappa - 2v^3)h^3 + 2v^3 y_j h^2 + (1-\mu)k(\pm 2(v^3\kappa - \kappa^2\theta v \\
&+ \kappa^2 v^2 y_j)h^3 + (4\kappa^2 v^2 {y_j}^2 - (2v^2 + 8\kappa\theta)\kappa v y_j + 2\kappa\theta(2\kappa\theta - v^2) - 2v^4)h^2 \\
&\pm((-8v^3\kappa + 4v^4\rho){y_j}^2 + (8\kappa\theta v^2 - 8v^3\rho r)y_j)h + (-8v^4\rho^2 + 8v^4){y_j}^2), \\
\zeta_{5,7} =& v^3\rho y_j h^2 + (1-\mu)k((v^3 {y_j}^2\kappa - v(v\kappa\theta + 2r\kappa v + \kappa v^2\rho)y_j \\
&- v(v^2 r - 2r\kappa\theta + v^3\rho))h^2 \pm(-v(2v^3\rho + v^3 + 4\kappa v^2\rho + 2v^2\kappa){y_j}^2 \\
&+ v(2v\kappa\theta + 4v\kappa\theta\rho + 4v^2\rho r + 2v^2 r)y_j)h + v(2v^3 + 6v^3\rho + 4v^3\rho^2){y_j}^2), \\
\zeta_{6,8} =& -v^3\rho y_j h^2 + (1-\mu)k((-v^3 {y_j}^2\kappa + v(v\kappa\theta + 2r\kappa v + \kappa v^2\rho)y_j \\
&+ v(v^2 r - 2r\kappa\theta + v^3\rho))h^2 \pm(v(-2v^3\rho + v^3 + 4\kappa v^2\rho - 2v^2\kappa){y_j}^2 \\
&+ v(2v\kappa\theta - 4v\kappa\theta\rho + 4v^2\rho r - 2v^2 r)y_j)h + v(2v^3 - 6v^3\rho + 4v^3\rho^2){y_j}^2).
\end{aligned}
$$

Numerical analysis: von Neumann stability

Rewrite $u^n_{i,j}$ as $u^n_{i,j} = g^n e^{Iiz_1+Ijz_2}$ where I is the imaginary unit, g^n is the amplitude at time level n, and $z_1 = 2\pi h/\lambda_1$ and $z_2 = 2\pi h/\lambda_2$ are phase angles with wavelengths λ_1 and λ_2, in the range $[-\pi, \pi]$, respectively

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$$|G|^2 \leq 1$$

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$\rightarrow$ consider case $r = \rho = 0$ and $\mu = 1/2$

Numerical analysis: von Neumann stability

Rewrite $u_{i,j}^n$ as $u_{i,j}^n = g^n e^{Iiz_1 + Ijz_2}$ where I is the imaginary unit, g^n is the amplitude at time level n, and $z_1 = 2\pi h/\lambda_1$ and $z_2 = 2\pi h/\lambda_2$ are phase angles with wavelengths λ_1 and λ_2, in the range $[-\pi, \pi]$, respectively
$\rightarrow$ scheme is stable if for all z_1 and z_2 the amplification factor $G = g^{n+1}/g^n$ satisfies

$$|G|^2 \leq 1$$

We would need to study polynomials in 13 variables...
$\rightarrow$ consider case $r = \rho = 0$ and $\mu = 1/2$

Theorem (B.D./Fournié '12)

For $r = \rho = 0$ and $\mu = 1/2$ (Crank-Nicolson), the scheme is unconditionally stable (von Neumann).

Stability analysis: sketch of the proof

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Define new variables

$$c_1 = \cos\left(\frac{z_1}{2}\right), \quad c_2 = \cos\left(\frac{z_2}{2}\right), \quad s_1 = \sin\left(\frac{z_1}{2}\right), \quad s_2 = \sin\left(\frac{z_2}{2}\right)$$

$$W = -\frac{2 s_2 s_1 \left(-\theta + v y\right)}{v}, \quad V = \frac{2 v y}{\kappa},$$

which allow us to express G in terms of h, k, κ, V, W and trigonometric functions only

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Stability validation for $\rho \neq 0$

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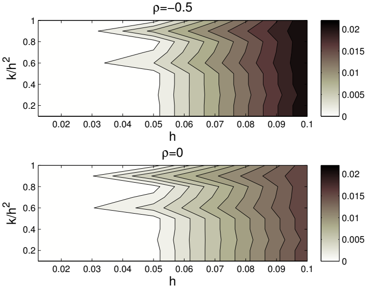

- little or no dependence of the error on the parabolic mesh ratio k/h^2
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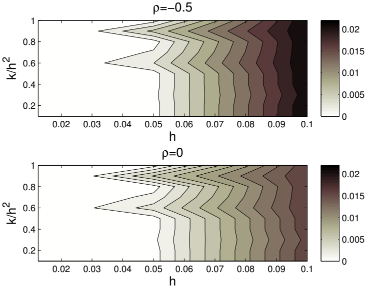

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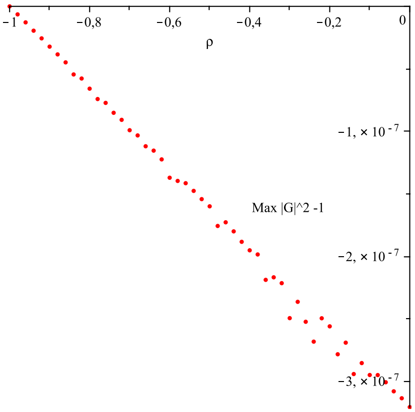
$\rightarrow$ conjecture that scheme is unconditionally stable and convergent also for general choice of parameters

Amplification factor for $\rho \neq 0$



- fix v, κ, θ to practical relevant values and replace all $\sin$ terms by equivalent $\cos$ expressions
- stability condition depends on ρ and c_1, c_2, y, h, k
- line-search global-optimization algorithm based on the Powell's and Brent's methods

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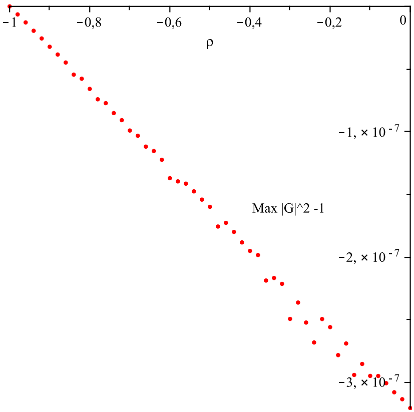

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- line-search global-optimization algorithm based on the Powell's and Brent's methods

$\rightarrow$ maxima for each ρ are always negative and very close to zero ($|G|^2 = 1$ for $y = 0$)

$\rightarrow$ conjecture that stability condition is satisfied although hard to prove analytically

Numerical efficiency and convergence

We use the parameters

$$K = 100,\ T = 0.5,\ r = 0.05,\ v = 0.1,\ \kappa = 2,\ \theta = 0.01,\ \rho = -0.5$$

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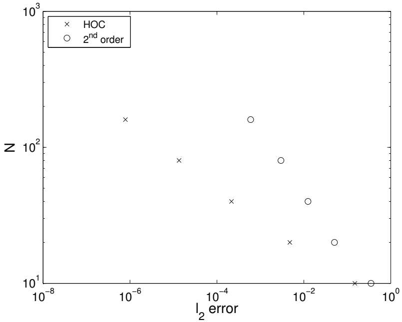

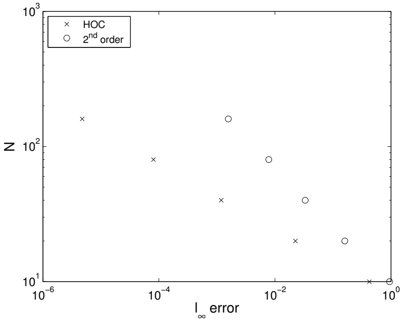

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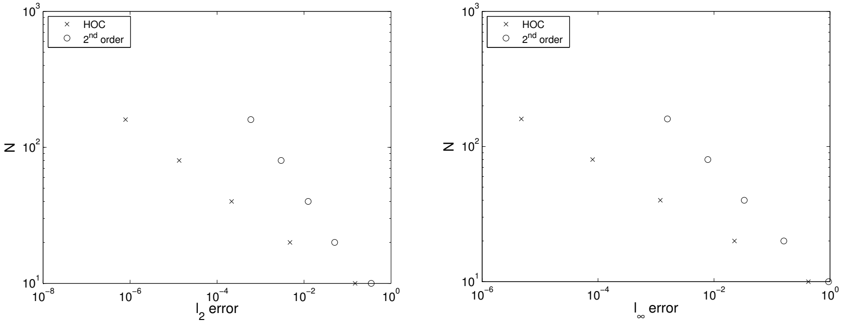

$\rightarrow$ for similar computational effort: orders of magnitude better in error

$\rightarrow$ to achieve a given error level: order of magnitude less computational effort

Non-uniform grids

Goal: concentrate grid points around strike K
$\rightarrow$ introduce transformation φ from non-uniform to uniform grid:

$$\hat{S} = \ln(S/K), \quad \varphi(x) = \hat{S}$$

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B.D., M. Fournié and C. Heuer.
High-order compact finite difference schemes for option pricing in stochastic volatility models on non-uniform grids.
J. Comput. Appl. Math. **271**, 2014. (arXiv:1504.5138)

High-order ADI schemes

Consider convection-diffusion equation

$$u_t = \operatorname{div}(D\nabla u) + c \cdot \nabla u$$

on a rectangular domain $\Omega \subset \mathbb{R}^2$, supplemented with initial and boundary conditions with

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

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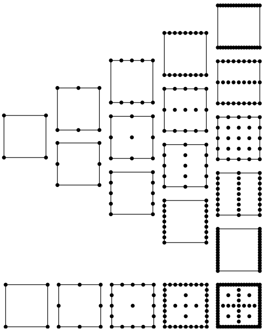
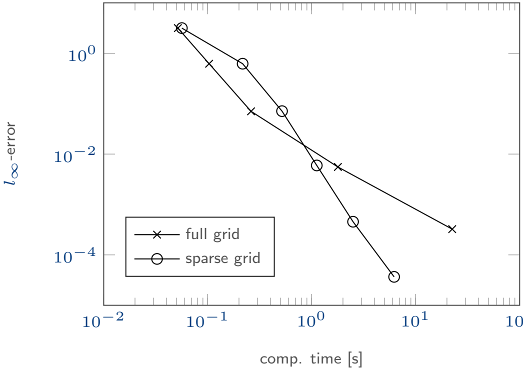
- Time: Hundsdorfer (2002) ADI, 2nd order in time
- Space: HOC scheme, 4th order

B.D., M. Fournié, A. Rigal.
High-order ADI schemes for convection-diffusion equations with mixed derivative terms.
In: *Spectral and High Order Methods for PDEs*, M. Azaïez et al. (eds.), LNCSE 95, Springer, 2013. (arXiv:1505.07621)

Sparse grid combination technique

- Further efficiency gains with sparse grids approach





B.D., C. Hendricks, J. Miles.
Sparse grid high-order ADI scheme for option pricing in stochastic volatility models. In: Novel Methods in Computational Finance, M. Ehrhardt et al. (eds.), pp. 295-312, Springer, 2017.

Partial-integro differential equation: Bates model

- additionally allow jumps in process for underlying asset
- pricing PIDE with additional (nonlocal) integral term
- implicit-explicit high-order scheme [cf. Salmi *et al.* '14]

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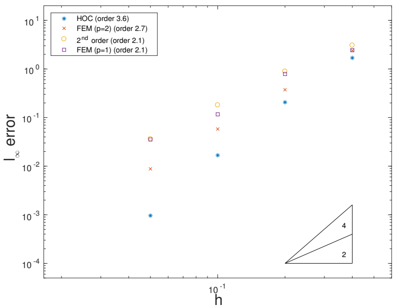

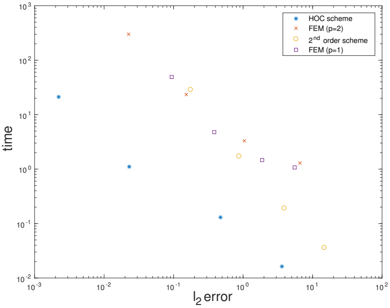

B.D. and A. Pitkin.
High-order compact finite difference scheme for option pricing in stochastic volatility jump models.
J. Comput. Appl. Math. **355**, 201-217, 2019. (arXiv:1704.05308)

Memory requirements: HOC vs. FD vs. FEM

Scheme	h	DOF	l_2-error	l_∞-error	Time (s)	Memory (kB)
HOC	0.4	121	3.6201	1.6891	0.016	6916
	0.2	441	0.4728	0.2063	0.130	+1060
	0.1	1681	0.0230	0.0168	1.106	+5536
	0.05	6561	0.0022	0.0009	21.145	+18284
FEM ($p = 2$)	0.4	441	6.5837	2.3944	1.294	123128
	0.2	1681	1.0438	0.3737	3.304	+1780
	0.1	6561	0.1522	0.0581	23.426	+8268
	0.05	25921	0.0225	0.0088	300.019	+40828
FD	0.4	121	14.8087	3.0653	0.036	6948
	0.2	441	3.9321	0.8913	0.191	+1772
	0.1	1681	0.8751	0.1806	1.715	+8384
	0.05	6561	0.1758	0.0364	28.706	+23064
FEM ($p = 1$)	0.4	121	5.5209	2.4373	1.072	123276
	0.2	441	1.8816	0.7876	1.462	+192
	0.1	1681	0.3846	0.1166	4.727	+2052
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$\rightarrow$ HOC very parsimonious, achieves high-order convergence without requiring additional unknowns, unlike finite element methods with higher polynomial order basis

High-order compact schemes: the price to pay

Drawbacks of high-order compact (HOC) schemes:

$\rightarrow$ derivation is algebraically demanding

$\rightarrow$ often ‘taylor-made’ for a specific application or rather small class of problems

$\rightarrow$ algebraic complexity is even higher in the numerical stability analysis

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Challenge: Can we generalize our HOC approach to a wider class of problems with mixed derivative terms?

Parabolic initial-boundary value problem

$$u_\tau + \sum_{i=1}^{n} a_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{\substack{i,j=1\\ i<j}}^{n} b_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} c_i \frac{\partial u}{\partial x_i} = g \quad \text{in } \Omega \times (0,T)$$

with initial condition $u_0 = u(x_1, \ldots x_n, 0)$ and boundary conditions, where $a_i = a_i(x_1, \ldots x_n, \tau) < 0$, $b_{ij} = b_{ij}(x_1, \ldots x_n, \tau)$, $c_i = c_i(x_1, \ldots x_n, \tau)$ and $g = g(x_1, \ldots x_n, \tau)$

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- HOC scheme: 4th order in space, 2nd order in time
- arbitrary spatial dimension
- stability analysis in 2D and 3D

B.D. and C. Heuer.
High-order compact schemes for parabolic problems with mixed derivatives in multiple space dimensions.
SIAM J. Numer. Anal. **53**(5), 2015. (arXiv:1506.06711)

Summary

- high-order compact (HOC) finite difference schemes for option pricing
- fourth-order in space, second-order in time
- thorough Fourier analysis: unconditional stability
- can be extended to non-uniform grids, HOC-ADI, sparse grids, PIDE, ...
- parsimonious in terms of memory requirements and computational effort, e.g. in comparison with finite element methods with higher polynomial order
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THANK YOU!