Asymptotic algebraic combinatorics

Alejandro Morales (University of Massachussetts at Amherst)* †
Igor Pak (University of California, Los Angeles)* †
Greta Panova (University of Southern California)* †
Dan Romik (University of California, Davis)* †
Nathan Williams (University of Texas at Dallas)†

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1 Introduction

Algebraic Combinatorics is an area of mathematics that employs methods of abstract algebra, notably group theory and representation theory, in various combinatorial contexts and, conversely, applies combinatorial techniques to problems in algebra and representation theory. Many of its problems arise from the need of quantitative and explicit understanding of algebraic phenomena like group representations and decompositions into irreducible representations, dimension formulae for modules, intersection numbers in geometry etc.

An important topic relates to Young tableaux, which carry the representation theory for the symmetric and general linear groups. The representation theory of S_n and GL_n is also carried with some structure to the ring of symmetric functions, which can also be interpreted as refined generating functions for discrete objects like Young Tableaux, plane partitions etc.

Initially Algebraic Combinatorics focused on the enumerative properties of these algebraic objects and then their combinatorial understanding. Notably such a result was the famous Hook-Length formula of Frame-Robbinson-Thrall, which gives the dimension of the irreducible representation $\mathbb{S}_\lambda$ of S_n, equivalently the number f^λ of Standard Young Tableaux of shape λ, as a product formula over the boxes in the Young diagram of λ:

$$f^\lambda = \dim \mathbb{S}_\lambda = \frac{n!}{\prod_{u\in\lambda} h_u}\,. \tag{1}$$

Besides its inherent beauty, this formula is useful in Asymptotic Representation Theory and Probability to determine the "typical" shape λ of a random standard Young Tableaux in the *Plancherel distribution*, a classical result of Vershik–Kerov [32] (see also the book [26] and the survey [29]).

Other such remarkable formulas include the enumeration of reduced decompositions of some permutations (see [4]), the hook-content formula for the number of semi-standard Young Tableaux (see e.g. [14, 28]). When no formula is possible, a concise combinatorial interpretation is also a solution like the rule for computing the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$, the multiplicities in the tensor product decompositions of GL_n irreducibles, giving them as the number of integer points in a polytope.

*Workshop co-organizer
†Report coauthor

However, more often than not, formulas like (1) and interpretations like the Littlewood-Richardson rule are a miracle rather than a property. No explicit product formula is known for the number of plane partitions of general shapes λ, nor for the number of skew SYTs and skew SSYTs nor for the number of reduced words of a permutation, nor for the number of monomials of a Schubert polynomial. No positive formula, nor any form of discrete interpretation (#P formula) is known for the Kronecker coefficients (the analogues of Littlewood–Richardson coefficients for the decomposition of tensor products of irreducible representations of the symmetric group), nor for the *Gromov–Witten invariants* for the cohomology of the Grassmannian (whose combinatorics study is yet another major subfield of Algebraic Combinatorics), or alternatively the multiplicative structure constants of *Schubert polynomials*.

Yet discrete objects of algebraic origins like Schur functions, SYTs and plane partitions are also related to integrable models of particle or dimer configurations within Statistical Mechanics. These objects are also central to Asymptotic Representation Theory. In these fields, and, more generally, in Probability, the problems are about understanding the large scale limit or asymptotic behavior rather than having explicit exact formulas and descriptions. Other problems, coming from Geometric Complexity Theory (GCT) and the distinction of permanent vs determinant as a "P vs NP" approach, involve the comparison of various representation-theoretic multiplicities related to the Littlewood-Richardson, plethysm and Kronecker coefficients.

Algebraic Combinatorics has long since reached the boundary of the universe of explicit exact formulas. Yet major problems from Statistical Mechanics to GCT remain unsolved as they don't fall into the special world of objects enumerated by exact product formulas. Hence we need to find asymptotic formulas for the main objects of Algebraic Combinatorics and study the emerging field of ***Asymptotic Algebraic Combinatorics***.

Previous advances in such directions came from Asymptotic Representation Theory originated by Vershik–Kerov, Kirillov, Okounkov, Borodin–Gorin–Rains, Biane et al. and continued recently in light of Integrable Probability and the analysis of interacting particle systems through interlacing arrays corresponding in the discrete world to SSYTs. Such results were largely possible thanks to the presence of "nice" formulas like product formulas, finite determinants, Selberg-type integrals etc, see [11, 12, 17].

Faced with the lack of such formulas for objects like skew SSYTs and Littlewood–Richardson coefficients, we now aim to study the asymptotics of these quantities using algebraic combinatorics, bijections, analytic combinatorics, and computer simulations. The workshop gathered specialists in Algebraic Combinatorics, Probability, Representation Theory with the aim of studying various available techniques for such analysis beyond the settings of integrable probability or classical analytic combinatorics.

2 Outstanding open problems

The main goal of the workshop was to bring together people from all the relevant areas, which are naturally very disjoint—Algebraic Combinatorics, Analytic Combinatorics, Probability, and Representation Theory, to share results and methods and establish the ***asymptotic study of objects and quantities in Algebraic Combinatorics***. As mentioned in the introduction, the asymptotic analysis is the next step after exhausting the possibilities for nice product formulas. Yet, it is also necessary for the applications into Statistical Mechanics, Asymptotic Representation Theory, Geometric Complexity Theory and Algebraic Geometry.

Below we include a list of some concrete problems and applications that we were hoping we might address and make progress on during and as a result of the workshop. It is worth noting that, between the time we made our plans for the workshop and its actual occurrence, several of the problems were solved; this illustrates the fast-paced and dynamic nature of the field.

- Determine the asymptotic expansion of $f^{\lambda/\mu}$ as $\lambda/\mu \to \alpha/\gamma$, a region in the plane between two curves α, γ. By [18] we have $\log f^{\lambda/\mu} = \frac{1}{2} n \log n + O(n)$, where recent work of Morales-Pak-Tassy show that the next term of the expansion is $cn + o(n)$ for some constant c. Determine its dependence on the limit shape.

- Study the asymptotic behavior of $f^{\lambda/\mu}$ when λ, μ approach some limit curves. The question for what limit curves of λ/μ with $|\mu| = c|\lambda|$ and $c > 0$ a fixed constant is the dimension $f^{\lambda/\mu}$ asymptotically maximal has been recently answered in [24].

- Determine the asymptotic growth of the Littlewood–Richardson coefficients $c^{\lambda}_{\mu\nu}$, when λ, μ, ν approach certain limit curves as before, under the right rescaling.
- Stanley showed in [28, supp. exc. 79(c)] that for the Littlewood–Richardson coefficients we have $\log_2 \max_{\lambda,\mu,\nu,\lambda\vdash n} c^{\lambda}_{\mu\,nu} \sim \frac{n}{2}$. Find the limit shapes of the partitions λ, μ, ν where this maximum is achieved, and determine the asymptotic behavior when $\updownarrow, \mu, \nu$ approximate given curves. The conjecture of Stanley claims that both λ, μ and ν have to be Plancherel for the maximum to be achieved (personal communication), and was partially resolved in [24].
- For the Kronecker coefficients $g(\lambda, \mu, \nu)$ what is $\log_2 \max_{\lambda,\mu,\nu\vdash n} g(\lambda, \mu, \nu)$, and what is the general asymptotic behavior when λ, μ, ν approach given curves. The maximum is achieved when λ, μ, ν are all Plancherel [24].
- For the plethysm coefficients $a_\lambda(d[n])$, relevant to GCT, determine their growth as a function of n, d, λ.
- Determine $\mathfrak{S}_w(1^n)$, the evaluation of the Schubert polynomial at $(1, \ldots)$, asymptotically as a function of n and certain characteristics of w like occurrence of 231 patterns, see [15, 30, 33].
- Determine the number R_w of reduced words of w asymptotically as a function of certain characteristics of w like occurrence of 231 patterns.
- Determine the typical permutations at the equator of the *weak Bruhat order*. This problem is equivalent to the "great circle conjecture" in the *random sorting networks* of [2], and was solved in an exciting recent paper by Duncan Dauvergne (who presented his results at the workshop; see Section 3 below).
- Study the asymptotic expansion as a function of the variables x of the normalized skew Schur function

$$\frac{s_{\lambda/\mu}(x_1, \ldots, x_k, 1^{n-k})}{s_{\lambda/\mu}(1^n)}$$

  as λ/μ approaches a limit profile. Alternatively, analyze lozenge tilings in "skew" domains, i.e. left and right boundaries given by μ, λ respectively, as opposed to the only studied case when one boundary is flat.
- Study the asymptotics of S_n normalized characters (after the breakthrough results by Biane, see [3], and Feray–Śniady, [6])
- Study the asymptotic limit surface of the distribution of monomials of Hilbert series in diagonal harmonics, proposed and christened "Marco Polo surface" by A. Garsia (cf. the proof of the *shuffle conjecture* in [5]).

3 Selected advances in the theory presented at the workshop

Twenty-two speakers presented their work during the workshop. We include below a short list of the theoretical advances presented in a few of the talks.

1. **Hook formulas for enumeration and asymptotics of skew tableaux**. Many analyses of the asymptotic behavior of Young tableaux are based on the classic hook-length product formula of Frame, Robinson and Thrall [7] for tableau of partition shape. However, this approach is limited in that it does not make it possible to answer questions about limit shapes except in a few simple cases (rectangular and staircase-shape tableaux) or derive asymptotics for tableaux of skew shapes, a regime where no product formula is known. Alejandro Morales (University of Massachussetts, Amherst) presented a novel enumerative and asymptotic analysis for skew shapes relying on a new beatiful class of formulas to count skew tableaux by Naruse [20, 21] (announced in 2014) and Okounkov–Olshanski (1998) [23] that are positive sums of products of hooks. This talk summarized several of his recent papers, coauthored with Pak and Panova, studying the combinatorics and asymptotics of Naruse's formula and an upcoming paper with Zhu on the Okounkov–Olshanski formula.

2. **The Archimedean limit of random sorting networks.** Duncan Dauvergne from the University of Toronto presented his recent proof of the 2006 random sorting networks conjecture made by Angel, Holroyd, Romik and Virág [2]. This exciting result is the culmination of a sequence of papers by the authors of the 2006 paper, Gorin, Mustazee, and others, and constitutes a major advance in the field. The methods developed to attack the problem will surely play a role in future work on offshoots and analogues of the original problem.

3. **Limit shapes of Young tableaux.** Svante Linusson (KTH Royal Institute of Technology, Stockholm) presented his recent joint work with Robin Sulzberger and Samu Potka on the limit shape of shifted staircase standard Young tableaux, with applications to 132-avoiding sorting networks. The results are an application of the calculus of variations methods used to analyze limit shapes of random Young tableaux, as developed in earlier work by Vershik-Kerov, Logan-Shepp and Pittel-Romik, and discussed in the book [26].

4. **Generalized Pólya urn models and Young tableaux.** Cyril Banderier (Université Paris-Nord) presented additional results on limit distributions for random staircase shape Young tableaux derived in his recent work with Marchal and Wallner. These results are based on an innovative technique that relates the questions on these tableaux to a class of generalized Pólya urns. The related questions on Pólya urns can in turn be answered using sophisticated methods of analytic combinatorics.

5. **Evaluation of determinants.** Many problems in asymptotic algebraic combinatorics lead to algebraic quantities such as determinants, which we often wish to evaluate precisely and/or asymptotically. This leads to difficult questions about evaluation of determinants. Christian Krattenthaler (University of Vienna) gave a survey talk in which he illustrated with a few examples how determinants with interesting evaluations can arise in applications, and how one can formulate conjectures about the evaluations of such determinants and eventually prove those conjectures through a series of artful manipulations, inspired generalizations, experimental calculations, and a carefully honed intuition.

4 Selected open problems presented at the workshop

In addition to presentations by speakers of recent advances in the theory and survey talks about established methods and directions, we held a session in which participants could present some exciting open problems they are thinking about. In this section we discuss just a few of the problems presented. A full list of problems was compiled by Nathan Williams, one of the workshop participants, and made available to all the participants. We hope that this will stimulate future research and collaboration efforts.

4.1 Constants in asymptotic expansions of tilings (presented by Vadim Gorin, MIT)

The number of tilings of a hexagonal domain Ω by lozenges was given by MacMahon as

$$\#\text{tilings of } \Omega = \prod_{i=1}^{A}\prod_{j=1}^{B}\prod_{k=1}^{C} \frac{i+j+k-1}{i+j+k-2}.$$

The number of tilings of more general domains by lozenges don't usually have such nice formulas, but we can still ask about the following asymptotic expansion:

$$\ln\big(\#\text{tilings of } L\cdot\Omega\big) = L^2\gamma_1 + (L\ln L)\gamma_2 + L\gamma_3 + \cdots . \tag{2}$$

It is known that the first constant γ_1 in Equation 2 can be written for any domain as

$$\gamma_1 = \iint_\Omega S(\nabla h)dxdy.$$

The second constant γ_2 is believed to be "somewhat universal," mostly not dependent on the shape of the domain.

Problem 4.1. *What is the third constant in Equation 2? Vadim thinks it should be able to be written as*

$$\gamma_3 = \oint_{\psi} f(\textit{tangent}) d\ell,$$

where f is some "mysterious" function.

4.2 Limit properties of tableau tuples (presented by Jang Soo Kim, Sungkyunkwan University)

Let A be a fixed standard Young tableau (SYT) of size r. We say that an SYT T *contains* A if A is equal to the tableau obtained from T by removing all cells containing integers greater than r.

Conjecture 4.2. *[10, Conjecture 6.1]*

$$\begin{aligned}\lim_{n\to\infty} \Pr &\begin{pmatrix} \textit{a random } m\textit{-tuple } (T_1,\dots,T_m) \textit{ of} \\ \textit{SYT of size } n\textit{, all the same shape} \\ \textit{with each } T_i \textit{ containing } A \end{pmatrix} \\ &= \lim_{n\to\infty} \Pr \begin{pmatrix} \textit{a random } m\textit{-tuple } (T_1,\dots,T_m) \textit{ of} \\ \textit{SYT of size } n\textit{,} \\ \textit{with each } T_i \textit{ containing } A \end{pmatrix}.\end{aligned}$$

This conjecture may seem intuitively obvious by the following argument. The only difference between the two limits is the condition that $T_1, \dots, T_m$ must have the same shape. Since there is a limit shape of a standard Young tableau of size n when n tends to infinity, even if we choose $T_1, \dots, T_n$ independently, the shapes of $T_1, \dots, T_m$ should be almost the same. Therefore, the additional condition should not change the probability, and we should obtain (**??**).

But this "intuitive proof" is incorrect because we haven't used any properties of tableau containment: if it were correct, it would hold for *any* property of tableaux. More generally, let $p : \text{SYT} \to \{\text{True}, \text{False}\}$ be a tableau property and define

$$P_n = \Pr \begin{pmatrix} \text{a random } m\text{-tuple } (T_1,\dots,T_m) \text{ of} \\ \text{SYT of size } n\text{, all the same shape} \\ p(T_i) \text{ for } 1 \le i \le m \end{pmatrix} \text{ and}$$

$$Q_n = \Pr \begin{pmatrix} \text{a random } m\text{-tuple } (T_1,\dots,T_m) \text{ of} \\ \text{SYT of size } n\text{,} \\ p(T_i) \text{ for } 1 \le i \le m \end{pmatrix}.$$

For example, when $p(T)$ is the statement that the number of rows of T is greater than the number of columns of T then

$$\frac{1}{2} = \lim_{n\to\infty} P_n \neq \lim_{n\to\infty} Q_n = \frac{1}{2^m}.$$

Problem 4.3. *Find the statements $p(T)$ for which $\lim_{n\to\infty} P_n = \lim_{n\to\infty} Q_n$.*

Note. *Fedor Petrov sketched the idea of a proof of Conjecture 4.2.*

4.3 Problems related to hook formulas for d-complete posets (presented by Soichi Okada, Nagoya University)

Theorem 4.4. *(Naruse–Okada [21]) Let P be a connected d-complete poset and F an order filter of P. Then the multivariate generating function of $(P \setminus F)$-partitions is given by*

$$\sum_{\pi \in \mathcal{A}(P\setminus F)} \boldsymbol{z}^{\pi} = \sum_{D \in \mathcal{E}_P(F)} \frac{\prod_{v\in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P\setminus D}(1 - \boldsymbol{z}[H_P(v)])},$$

where D runs over all excited diagrams of F in P.

Problem 4.5. *Find a combinatorial (bijective) proof. More explicitly, find a decomposition*

$$\mathcal{A}(P \setminus F) = \bigsqcup_{D \in \mathcal{E}_P(F)} \mathcal{A}(P \setminus F)_D$$

such that

$$\sum_{\pi \in \mathcal{A}(P \setminus F)_D} \boldsymbol{z}^\pi = \frac{\prod_{v \in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P \setminus D}(1 - \boldsymbol{z}[H_P(v)])}.$$

Note. *This is proven in the famous paper [16] for skew shapes using the Hillman-Grassl algorithm.*

Conjecture 4.6. *(Okada [22]) If P is a connected d-complete poset, then*

$$\sum_{\sigma \in \mathcal{A}(P)} W_P(\sigma; q, t) \boldsymbol{z}^\sigma = \prod_{v \in P} \frac{(t\boldsymbol{z}[H_P(v)]; q)_\infty}{(\boldsymbol{z}[H_P(v)]; q)_\infty}.$$

Known Results : The conjecture holds

- if $q = t$ (Peterson–Proctor's hook product formula),
- if P is a rooted tree (use the binomial theorem and induction),
- if $P = D(\lambda)$ is a shape (= Young diagram) (Okada),
- if $P = S(\lambda)$ is a shifted shape (= shifted Young diagram) (Okada),
- if P is a bird (Ishikawa).

4.4 Asymptotics of oscillating tableaux (presented by Sam Hopkins, University of Minnesota)

In his nice survey "The Ubiquitous Young Tableau" [27], Sagan highlights three classes of tableaux: ordinary Standard Young Tableaux, shifted Standard Young Tableaux, and oscillating tableaux. Many people have studied asymptotics of both ordinary and shifted SYTs—what about oscillating tableaux?

An SYT of shape λ is a walk in Young's lattice of partitions from the empty shape $\varnothing$ to λ that only goes upwards. An oscillating tableau of shape λ is a walk in Young's lattice of partitions from the empty shape $\varnothing$ to λ that can use both upwards and downwards steps. More formally, an *oscillating tableau of shape λ and length ℓ* is a sequence $(\varnothing = \lambda^0, \lambda^1, \dots, \lambda^\ell = \lambda)$ of partitions such that λ^i and λ^{i+1} differ in exactly one box for all i.

(Oscillating tableaux play roughly the same role for the symplectic group that SYT play for the general linear group.)

Problem 4.7. *Study asymptotics of oscillating tableaux. We can take either $\ell \to \infty$ or $|\lambda| \to \infty$ or both. In terms of what we "observe" from a random oscillating tableau $T = (\lambda^0, \lambda^1, \dots, \lambda^\ell)$, there are various options, including:*

- *observe the Young diagram $\lambda^{\lfloor c\ell \rfloor}$ for some $0 < c < 1$ (which hopefully limits to a limit curve with the right normalization);*
- *write i at a box u if u belongs to exactly i of the λ^j, and observe these heights as a surface (which hopefully limits to a limit surface with the right normalization).*

Why could we hope for nice asymptotics? Because there are equally good product formulas for oscillating tableaux as for SYT! Set $\mathrm{OT}(\lambda, \ell)$ to be the set of oscillating tableaux of shape λ and length ℓ.

Theorem 4.8 (Sundaram [31])**.** *Let $|\lambda| = k$. Then*

$$\#\mathrm{OT}(\lambda, k + 2n) = \binom{k + 2n}{k} \cdot (2n - 1)!! \cdot f^\lambda,$$

where $f^\lambda := \#\mathrm{SYT}(\lambda)$ is given by the usual hook-length formula.

There are also weights one can apply to oscillating tableaux (which don't make sense for SYTs) and which still give product formulas:

Theorem 4.9 (Hopkins-Zhang [9]). *Let* $|\lambda| = k$. *Then*

$$\sum_{T \in \mathrm{OT}(\lambda, k+2n)} \mathrm{wt}(T) = \#\mathrm{OT}(\lambda, k+2n) \cdot (k+2n+1) \cdot \frac{3k+2n}{6},$$

where $\mathrm{wt}(\lambda^0, \dots, \lambda^\ell) := \sum_{i=0}^{\ell} |\lambda^i|$.

Theorem 4.10 (Han-Xiong [8]). *Let* $|\lambda| = k$. *Then*

$$\sum_{T \in \mathrm{OT}(\lambda, k+2n)} \mathrm{wt}_P(T) = \#\mathrm{OT}(\lambda, k+2n) \cdot (k+2n+1) \cdot Q(k, 2n+k),$$

where $\mathrm{wt}_P(\lambda^0, \dots, \lambda^\ell) := \sum_{i=0}^{\ell} P(|\lambda^i|, i)$ *for any fixed bivariate polynomial* $P(x, y)$*, and* $Q(x, y)$ *is another bivariate polynomial depending on* P *(in a recursive way).*

It would also be interesting to study random oscillating tableaux according to these weights (e.g., can clearly obtain Plancherel measure with $\lambda = \varnothing$ in this way).

4.5 Nonstandard longest increasing subsequences (presented by Robin Pemantle, University of Pennsylvania)

It is well-known [26, 32] that the expected length of the longest increasing subsequence (LIS) of a random permutation of length n is $2\sqrt{n}$. Consider instead the expected length of the LIS of a simple random walk (i.e., steps of ± 1, each with probability $1/2$). By [1, Theorem 2], this expectation is at least $(1/1000)\sqrt{n}\log_2 n$.

The *Ultra-fat tailed random walk* is defined in [25, Section 2.2]. For a fixed number of steps, the following easier construction will do. Let π be a random permutation on $[N]$ and let $\{Y_i\}$ be IID ± 1 fair coin flips. For $k \le N$, let

$$S_k = \sum_{j=1}^{k} Y_j 10^{\pi(j)}.$$

Problem 4.11. *Determine the length of the LIS in* $S_1, S_2, \dots, S_N$.

It is shown in [25] that the mean and median are bounded between two powers of N, determined by recursions, and in particular it is at least $N^{0.69}$ and at most $N^{0.82}$. Neither recursion can yield a sharp bound. Simulations show it is around $N^{0.71}$.

4.6 Stanley's Schubert shenanigans (presented by Greta Panova, University of Southern California)

This problem originates with Richard Stanley. Each permutation $w \in S_n$ has a corresponding *Schubert polynomial* $\mathfrak{S}_w(x_1, x_2 \dots, x_n)$. Define

$$\gamma_n = \max_{w \in S_n} \mathfrak{S}_w(1, 1, \dots, 1).$$

Using well-known formulas for Schubert polynomials, γ_n can be rephrased in terms of reduced words, or in terms of pipe dreams.

Conjecture 4.12 ([30, Section 5]).

1. *Show that the limit* $\lim\limits_{n\to\infty} n^{-2} \log_2 \gamma_n$ *exists.*
2. *Compute the value* c *of this limit.*

3. *Find the permutation(s) w achieving γ_n.*

It is known that

$$2^{n^2/4} < \gamma_n < 2^{\binom{n}{2}}.$$

Moreover, if c exists, then $0.2932362762 < c < 0.46$; the lower bound is obtained by considering a family of "layered permutations" [19], while the upper bound follows from a "rigorous heuristic argument" due to due to Damir Yeliussizov and Igor Pak (unpublished) for the corresponding $\limsup$. In [15], Merzon and Smirnov conjectured (on the basis of numerical evidence for $n \leq 10$) that this family realizing the lower bound also realizes the upper bound:

Conjecture 4.13 ([15, Conjecture 5.7]). *Every permutation $w \in S_n$ with $\mathfrak{S}_w(1, 1, \ldots, 1) = \gamma_n$ is a layered permutation.*

5 A panel discussion on the future of the field

This workshop has been the first attempt by the organizers to gather together experts from multiple areas with the goal of catalyzing new research and a fruitful exchange of ideas in asymptotic algebraic combinatorics. We feel that the area of the workshop is undergoing tremendous growth and becoming established as an important area of research. To help inform our and the participants' thinking about future directions, we held a panel session in which the future of asymptotic algebraic combinatorics (and more generally algebraic combinatorics), was discussed. The panelists were Igor Pak and Dan Romik, and the panel was moderated by Greta Panova. Many others among the workshop participants expressed opinions about future directions for the field, both in the mathematical sense and in terms of how activities should be organized, how we can better support the progress of graduate students and early-career researchers, and so on. This was a thought-provoking experience.

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